

# CAUCHY PROBLEM FOR EFFECTIVELY HYPERBOLIC OPERATORS WITH TRIPLE CHARACTERISTICS OF VARIABLE MULTIPLICITY

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**ABSTRACT.** We study a class of third order hyperbolic operators  $P$  in  $G = \{(t, x) : 0 \leq t \leq T, x \in U \subset \mathbb{R}^n\}$  with triple characteristics on  $t = 0$ . We consider the case when the fundamental matrix of the principal symbol of  $P$  for  $t = 0$  has a couple of non-vanishing real eigenvalues. Such operators are called effectively hyperbolic. V. Ivrii introduced the conjecture that every effectively hyperbolic operator is *strongly hyperbolic*, that is the Cauchy problem for  $P + Q$  is locally well posed for any lower order terms  $Q$ . This conjecture has been solved for operators having at most double characteristics. A strongly hyperbolic operator in  $G$  could have triple characteristics in  $G$  only for  $t = 0$  or for  $t = T$ . We prove that the operators in our class are strongly hyperbolic if  $T$  is small enough. Our proof is based on energy estimates with a loss of regularity.

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## 1. INTRODUCTION

**1.1. Introduction and main result.** Consider a differential operator

$$P(t, x, D_t, D_x) = \sum_{\alpha+|\beta| \leq m} c_{\alpha,\beta}(t, x) D_t^\alpha D_x^\beta, \quad D_t = -i\partial_t, D_{x_j} = -i\partial_{x_j} \quad (1.1)$$

of order  $m$  with  $C^\infty$  coefficients  $c_{\alpha,\beta}(t, x)$ ,  $t \in \mathbb{R}$ ,  $x \in \mathbb{R}^n$ . Denote by

$$p_m(t, x, \tau, \xi) = \sum_{\alpha+|\beta|=m} c_{\alpha,\beta}(t, x) \tau^\alpha \xi^\beta$$

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the principal symbol of  $P$ . We assume that  $c_{m,0}(t, x) \neq 0$  for all  $(t, x)$ . Let  $\Omega \subset \mathbb{R}^{n+1}$  be an open set and let

$$\Omega_\eta^- = \Omega \cap \{t \leq \eta\}, \Omega_\eta^+ = \Omega \cap \{t \geq \eta\}, G = \Omega \cap \{0 \leq t \leq T\}.$$

Set  $P_m(t, x, D_t, D_x) = p_m(t, x, D_t, D_x)$ .

**Definition 1.1.** *We say that the Cauchy problem*

$$Pu = f \text{ in } \Omega \cap \{t < T\}, \text{ supp } u \subset \overline{G} \quad (1.2)$$

*is well posed in  $G$  if*

- (i) (existence) *for every  $f \in C_0^\infty(\Omega)$ ,  $\text{supp } f \subset \overline{\Omega_0^+}$  there exists a solution  $u \in \mathcal{D}'(\Omega)$  satisfying (1.2).*
- (ii) (uniqueness) *if  $u \in \mathcal{D}'(\Omega)$  satisfies (1.2), then for every  $s, 0 < s \leq T$ , if  $Pu = 0$  in  $\Omega_s^-$ , then  $u = 0$  in  $\Omega_s^-$ .*

A necessary condition for the well posedness of the Cauchy problem (WPC) is the hyperbolicity of the operator  $P$  in  $G$  (see [7] and the references cited there). This means that for every  $(t_0, x_0, \xi) \in G \times \mathbb{R}^n \setminus \{0\}$  the equation

$$p_m(t_0, x_0, \tau, \xi) = 0 \quad (1.3)$$

with respect to  $\tau$  has only real roots  $\tau = \lambda_j(t_0, x_0, \xi)$ .

**Definition 1.2.** *We say that the operator  $P$  with principal symbol  $p_m$  is strongly hyperbolic in  $G$  if for every point  $z_0 = (t_0, x_0) \in G$  there exists a neighborhood  $U$  of  $z_0$  and  $T_0 \geq 0$  ( $T_0 < T$  if  $t_0 = T$  and  $T_0 = 0$  if  $t_0 = 0$ ) such that the Cauchy problem (1.2) for the operator  $L = P_m(t, x, D_t, D_x) + Q_{m-1}(t, x, D_t, D_x)$  is well posed in  $U_s^+$  for every  $T_0 \leq s < T(U)$  and for any operator  $Q_{m-1}(t, x, D_t, D_x)$  of order less or equal to  $m - 1$ .*

When  $P$  is strictly hyperbolic, that is when the equation (1.3) has simple roots  $\lambda_j(t, x, \xi)$  with respect to the variable  $\tau$  for all  $(t, x, \xi) \in G \times \mathbb{R}^n \setminus \{0\}$ , it is a classical result that  $P$  is strongly hyperbolic. If the equation (1.3) has real roots with constant multiplicity for  $(t, x, \xi) \in G \times \mathbb{R}^n \setminus \{0\}$ , the operator  $P$  is strongly hyperbolic **if and only if** it is strictly hyperbolic. Thus in the case of roots with constant multiplicity—greater than 1—we must impose conditions on the lower order terms  $Q_{m-1}$ , called Levi conditions, in order that the Cauchy problem be well posed. The analysis of the Cauchy problem for such operators is complete and we know the necessary [4] and sufficient [3] conditions for (WPC).

Passing to the case when the roots of (1.3) have variable multiplicity, notice that the roots  $\lambda_j(t, x, \xi)$  in general are not smooth but only continuous. The case of operators with constant coefficients is also completely examined and  $P$  is strongly hyperbolic **if and only if**  $P$  is strictly hyperbolic. The necessary and sufficient condition of Gårding for (WPC) says that there exists a constant  $c > 0$  such that for the full symbol  $p(\tau, \xi)$  of  $P$  we have

$$p(\tau, \xi) \neq 0, \text{ for } |\text{Im } \tau| > c, \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

In the following we switch to a different notation for the sake of simplicity and denote  $t = x_0$ ,  $x = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ . The dual variables are denoted by  $\xi = (\xi_0, \xi_1, \dots, \xi_n) = (\xi_0, \xi')$ .

Given a symbol  $p(x, \xi)$ , let

$$\Sigma(p) = \{z \in T^*G \setminus \{0\} : p(z) = 0\}, \Sigma_1(p) = \{z \in T^*(G) \setminus \{0\} : p(z) = 0, dp(z) = 0\}.$$

In the case  $\Sigma_1(p_m) = \emptyset$ , the operator is of principal type and a hyperbolic operator  $P$  in  $G$  is strongly hyperbolic (see Section 23.4 in [6]).

Passing to the case  $\Sigma_1(p_m) \neq \emptyset$ , notice that if we have a critical point  $(\hat{x}, \hat{\xi}) \in \Sigma_1(p)$ , then the Hamiltonian system

$$\frac{dx}{ds} = \partial_\xi p, \quad \frac{d\xi}{ds} = -\partial_x p$$

has a stationary point and it is natural to consider the differential of the Hamilton vector field. Thus we are led to define the fundamental matrix

$$F_p(\hat{x}, \hat{\xi}) = \begin{pmatrix} p_{\xi,x}(\hat{x}, \hat{\xi}) & p_{\xi,\xi}(\hat{x}, \hat{\xi}) \\ -p_{x,x}(\hat{x}, \hat{\xi}) & -p_{x,\xi}(\hat{x}, \hat{\xi}) \end{pmatrix}.$$

We recall two important properties of  $F_p$  (see [7], [5]):

1. For every point  $z \in \Sigma_1(p)$  the Hessian  $Q_p(X, Y)$ ,  $X, Y \in T_z(T^*(G))$  at  $z$  of  $\frac{p}{2}$  is well defined. Then  $Q_p(X, Y) = \sigma(X, F_p(z)Y)$ ,  $\sigma$  being the symplectic form on  $T^*(G)$ . Thus after a canonical transformation the fundamental matrix is transformed into a similar one and its eigenvalues are invariant under canonical transformations. Hörmander [5] called  $F_p(z)$  the Hamilton map of  $Q_p$ .
2. If  $P$  is hyperbolic in  $G$  and  $(\hat{x}, \hat{\xi})$  is a critical point of  $p_m(x, \xi)$ , then  $F_{p_m}(\hat{x}, \hat{\xi})$  has at most two non-vanishing real simple eigenvalues  $\mu$  and  $-\mu$  and all other eigenvalues  $\mu_j$  are purely imaginary, that is  $\text{Re } \mu_j = 0$ .

The existence of non-vanishing real eigenvalues of  $F_{p_m}(\hat{x}, \hat{\xi})$  is a *necessary condition* for strong hyperbolicity. More precisely, let  $p_{m-1}(x, \xi) = \sum_{|\alpha|=m-1} c_\alpha(x) \xi^\alpha$  and let

$$p'_{m-1}(x, \xi) = p_{m-1}(x, \xi) + \frac{i}{2} \sum_{j=0}^n \frac{\partial^2 p_m}{\partial x_j \partial \xi_j}(x, \xi)$$

be the subprincipal symbol of  $P$  which is invariantly defined for  $(x, \xi) \in \Sigma_1(p_m)$ . Then we have the following

**Theorem 1.1** (Theorem 3 and Corollary 3 in [7]). *If  $P$  is strongly hyperbolic in  $G$ , then at every point  $(\hat{x}, \hat{\xi}) \in \Sigma_1(p_m)$  the fundamental matrix  $F_{p_m}(\hat{x}, \hat{\xi})$  has two non-vanishing real eigenvalues. Moreover, for  $(x, \xi') \in \overset{\circ}{G} \times (\mathbb{R}^n \setminus \{0\})$  the multiplicities of the roots of (1) are not greater than two, and for  $(x, \xi') \in \{x_0 = 0\} \times \mathbb{R}^n \setminus \{0\}$  or for  $(x, \xi') \in \{x_0 = T\} \times \mathbb{R}^n \setminus \{0\}$  these multiplicities are not greater than three. If  $F_{p_m}(\hat{x}, \hat{\xi})$  has only purely imaginary eigenvalues, the condition  $\text{Imp}'_{m-1}(\hat{x}, \hat{\xi}) = 0$  is necessary for (WPC).*

If  $F_{p_m}(\hat{x}, \hat{\xi})$  has only purely imaginary eigenvalues, for (WCP) we have a second necessary condition

$$|\text{Re } p'_{m-1}(\hat{x}, \hat{\xi})| \leq \frac{1}{4} \sum_{j=0}^{2n+2} |\mu_j|,$$

$\mu_j$  being the eigenvalues of  $F_{p_m}(\hat{x}, \hat{\xi})$  repeated according to their multiplicities. This condition has been proved in [7] in some special cases concerning the structure of  $F_{p_m}(\hat{x}, \hat{\xi})$  and without any restriction by Hörmander [5].

**Definition 1.3.** *A hyperbolic operator with principal symbol  $p_m(x, \xi)$  will be called effectively hyperbolic if at every point  $(\hat{x}, \hat{\xi}) \in \Sigma_1(p_m)$ , the fundamental matrix  $F_{p_m}(\hat{x}, \hat{\xi})$  has two non-vanishing real eigenvalues.*

V. Ivrii introduced the following

**Conjecture.** *A hyperbolic operator is strongly hyperbolic if and only if it is effectively hyperbolic.*

For operators with at most double characteristics the sufficient part of the above conjecture has been established for some special class of effectively operators by Oleinik [15], Hörmander [5], Ivrii [8], Melrose [11] and in the general case by N. Iwasaki [9], [10] and T. Nishitani [12], [13], [14]. In particular, in the works of Nishitani many properties of effectively hyperbolic operators with double characteristics have been established. An important phenomenon for this class of operators is that we have always a loss of regularity which depends on the fraction of the subprincipal symbol and the non-vanishing real eigenvalue of  $F_{p_m}$  at double characteristic points (see Theorem 3 in [7] for a necessary condition and (1.7) below).

However according to Theorem 1.1, there are also classes of effectively hyperbolic operators with characteristics of multiplicity 3 which should be strongly hyperbolic. To our best knowledge, in the literature there are no examples of operators with triple characteristics which are strongly hyperbolic. The analysis of operators with double characteristics is fairly complete and over the last 30 years a lot of works treating both the effectively hyperbolic and the non-effectively hyperbolic operators has appeared.

The purpose of the present paper is to show that a class of third order weakly hyperbolic operators of the form (1.1) are strongly hyperbolic if their principal symbol has roots of multiplicity 3 on the points on the surface  $t = 0$ , where the Cauchy data are assigned and, moreover, if the Hamilton map  $F_{p_3}$  associated to the principal symbol  $p_3$  on such points has two real non-vanishing eigenvalues in its spectrum. According to Theorem 1.1, a strongly hyperbolic operator may have triple characteristics only for  $t = 0$  or  $t = T$  and in this paper we deal with the case when this happens for  $t = 0$

An example of "constant coefficients" model for such a class is the operator

$$P(t, D_t, D_x) = D_t^3 + ta_1(D_x)D_t^2 - ta_2(D_x)D_t + t^2a_3(D_x) + b_2(D_x). \quad (1.4)$$

Here  $a_j(D_x)$  denotes an operator with constant coefficients in the space variables  $x$  of order  $j$ ,  $j = 1, 2, 3$ , while  $b_2(D_x)$  is a second order operator. The coefficients  $t$  and  $t^2$  in front of  $a_1, a_2$  and  $a_3$  are not a restriction as our argument in Section 2 shows.

It is easy to see that the operator  $P$  is effectively hyperbolic if the symbol  $a_2(\xi)$  is *elliptic and positive*, i.e.  $a_2(\xi) \geq c|\xi|^2$ ,  $c > 0$ . Then the Hamilton map  $F_{p_3}$  of  $p_3$  has two real eigenvalues  $\pm a_2(\xi) \neq 0$ . Finally, by hyperbolicity, the symbols  $a_1(\xi)$  and  $a_3(\xi)$  must be real. In [1] we investigated the operator (1.4) and we proved that it is strongly hyperbolic with a loss of regularity related to  $\sup_{|\xi|=1} \left| \frac{p_2'(0,0,\xi)}{a_2(\xi)} \right|$ . Of course, for this example we may apply the Fourier transform with respect to  $x$ , but, to obtain a suitable a priori estimate, we need to work with a weight  $f^{-N} = (t/3 + \langle \xi \rangle^{-2/3})^{-N}$  with  $N$  depending on  $\max_{|\xi|=1} \left| \frac{p_2'(0,0,\xi)}{a_2(\xi)} \right|$ . The corresponding energy estimates have a  $2N/3 - 2$  loss of regularity. This was the first known example for effectively hyperbolic operator with triple characteristics which is strongly hyperbolic.

In this paper we study a class of hyperbolic operators with triple characteristics and variable coefficients which are effectively hyperbolic on the set  $t = 0$ . More precisely, we examine operators

having the form

$$\begin{aligned} P = & D_t^3 + q_1(t, x, D_x)D_t^2 + q_2(t, x, D_x)D_t + q_3(t, x, D_x) \\ & + r_2(t, x, D_x) + r_1(t, x, D_x)D_t + r_0(t, x)D_t^2 + m_1(t, x, D_x) + m_0(t, x)D_t + c_0(t, x). \end{aligned} \quad (1.5)$$

Here  $q_j(t, x, D_x)$ ,  $j = 1, 2, 3$ , are differential operators with  $C^\infty$  coefficients and real-valued symbols  $q_j(t, x, \xi)$  which are homogeneous polynomials of order  $j$  in  $\xi$ ,  $r_j(t, x, D_x)$ ,  $j = 1, 2$ , are differential operators with  $C^\infty$  coefficients and symbols  $r_j(t, x, \xi)$  homogeneous of order  $j$  with respect to  $\xi$ ,  $r_0(t, x)$ ,  $m_0(t, x)$ ,  $c_0(t, x)$  are  $C^\infty$  functions and  $m_1(t, x, D_x)$  is a first order differential operator with  $C^\infty$  coefficients. Let  $p_3(t, x, \tau, \xi)$  be the principal symbol of  $P$  and let  $G = \{(t, x) : 0 \leq t \leq T, x \in U\}$ , where  $U \subset \subset \mathbb{R}^n$  is an open set in  $\mathbb{R}^n$ . We suppose that  $P$  has  $C^\infty$  coefficients in a small neighborhood of  $\bar{G}$  and we make the following assumptions:

( $H_0$ ) The roots of the equation  $p_3(t, x, \tau, \xi) = 0$  with respect to  $\tau$  are real for all  $(t, x) \in \bar{G}$ ,  $\xi \in \mathbb{R}^n$ .

( $H_1$ ) The equation  $p_3(0, x, \tau, \xi) = 0$  with respect to  $\tau$  for  $t = 0$  has a triple root  $\tau = \lambda(x, \xi)$  for  $x \in \bar{U}$ ,  $\xi \in \mathbb{R}^n$ .

( $H_2$ ) The Hamiltonian map  $F_{p_3}(0, x, \lambda(x, \xi), \xi)$  of  $p_3$  has non-zero real eigenvalues  $\mu(x, \xi)$  for  $(x, \xi) \in \bar{U} \times \mathbb{R}^n$ .

It is clear that if we have a triple root for  $t = 0$ , then this root must have the form  $\lambda(x, \xi) = -\frac{1}{3}q_1(0, x, \xi)$ . Under the hypothesis ( $H_0$ ) – ( $H_1$ ) for the principal symbol  $p_3$ , we can change the variables preserving the manifold  $t = 0$  and reduce the analysis to the case when the principal symbol  $p_3$  for  $t = 0$ ,  $x \in \bar{U}$ ,  $\xi \in \mathbb{R}^n$  has a triple root  $\tau = 0$  (see Section 2). Our main result is the following

**Theorem 1.2.** *Assume the hypothesis ( $H_0$ ) – ( $H_2$ ) satisfied. Let  $G = \{(t, x) : 0 \leq t \leq T, x \in U\}$ . Then for  $T > 0$  sufficiently small, the operator  $P$  is strongly hyperbolic in  $G$ .*

Notice that if  $T$  is not small and if for  $0 < \delta \leq t < T$  with sufficiently small  $\delta$  the operator  $P$  is effectively hyperbolic with double characteristics in some points, we can combine our result with those of [9], [10], [12], [13] to obtain a strongly hyperbolic operator in  $\{0 \leq t < T, x \in U\}$ . Our arguments work also if we assume that  $q_1, q_2, q_3$  are pseudodifferential operators with real-valued symbols.

To prove Theorem 1.2, we establish a semi global theorem of the existence and uniqueness of a solution of the Cauchy problem (see Theorem 8.3). For this purpose we obtain an a priori estimate with a loss of regularity of order  $2N/3 - 2$  for the operator  $P$  having triple characteristics for  $t = 0$ . We choose  $N = 3\sqrt{3}\Pi + N_0$ , with  $N_0$  an integer and

$$\Pi = \frac{1}{3} + \sup_{x \in \bar{U}, |\xi|=1} |p_2'(0, x, \lambda(x, \xi), \xi)(\mu(x, \xi))^{-1}|, \quad (1.6)$$

where  $p_2'(t, x, \tau, \xi)$  is the subprincipal symbol of  $P$ . Moreover, the integer  $N_0$  depends only on  $p_3(0, x, \lambda(x, \xi), \xi)$  but we are not going to precise the optimal value of  $N_0$ . It seems that with a more complicated analysis of the contribution of the subprincipal symbol  $p_2'$  it should be possible to obtain a loss of regularity  $2N/3 - 2$  with  $N = \frac{3}{2}\Pi + N_0$  and this is an interested open problem.

We may compare the number (1.6) with the loss of regularity for second order strongly hyperbolic operators  $L$  with principal symbol  $l_2(t, x, \tau, \xi)$  given by

$$2 + \left\lceil \frac{1}{2} + \sup_{\rho \in \Sigma_1} |l'_1(\rho)(\mu(\rho)^{-1})| \right\rceil,$$

where  $[z]$  is the integer part of  $z$  (see [15], [8], [11] for a special class of operators with double characteristics and [13] for the general case). Here

$$\Sigma_1 = \{\rho = (t, x, \tau, \xi) \in G \times \mathbb{R}^{n+1} \setminus \{0\} : l_2(\rho) = 0, dl_2(\rho) = 0\}$$

is the double characteristic set of  $L$ ,  $l'_1(\rho)$  is the subprincipal symbol of  $L$  and  $\mu(\rho)$  is the non-vanishing eigenvalue of the Hamiltonian map  $F_{l_2}(\rho)$  at  $\rho \in \Sigma_1$ . It is important to note that the loss of regularity  $M$  for the solutions of the Cauchy problem for  $P$  is **bounded** from below by

$$\sup_{x \in \bar{U}, |\xi|=1} |\operatorname{Im}(p'_2(0, x, \lambda(x, \xi), \xi))(\mu(x, \xi))^{-1}| \leq 2n(M + 3). \quad (1.7)$$

This follows from the necessary condition (36) in Theorem 3 in [7]. Thus our result with  $M = 2N/3 - 2$  is compatible with this lower bound.

**1.2. Comments on the proof of the main result.** The proof of Theorem 1.2 is long and technical. It is based on the energy estimates obtained in Theorems 8.1 and 8.2 and, as we mentioned above, we cannot avoid the loss of regularity which is related to the fraction of the subprincipal symbol and the non-vanishing eigenvalue of the Hamiltonian map. This is one of the main differences compared to the hyperbolic operators of principal type (see for example the analysis in Section 23.4 in [6].)

First by a change of variables  $(t, x)$  we reduce the analysis to the case when the triple root of  $p_3$  for  $t = 0$  becomes  $\tau = 0$ . Thus in the new variables, which we denote again by  $(t, x)$ , the principal symbol has the form

$$p_3(t, x, \tau, \xi) = \tau^3 + ta_1(t, x, \xi)\tau^2 - ta_2(t, x, \xi)\tau + t^2a_3(t, x, \xi),$$

where  $a_j(t, x, \xi)$ ,  $j = 1, 2, 3$ , are real valued homogeneous polynomials with respect to  $\xi$  and  $a_2(t, x, \xi) \geq \delta|\xi|^2$  (see Section 2). Next, we introduce, in Section 4, the scaling  $t = \varepsilon^{2/3}s$ ,  $x = \varepsilon y$ ,  $\varepsilon > 0$ , and we transform our operator into the operator  $\mathcal{P}$  with respect to  $(s, y)$  (see Section 4 for the notations). Since we are interested in showing that the Cauchy problem is well posed for sufficiently small  $t$  and since  $P$  is strictly hyperbolic for  $t$  positive and small enough, we can investigate the operator  $\mathcal{P}$ . The symbols  $a_j(t, x, \xi)$  are transformed to symbols

$$a_j^\varepsilon(t, x, \xi) = a_j(\varepsilon^{2/3}t, \varepsilon x, \xi)$$

and this is important for the pseudodifferential calculus developed in Sections 3-4. Eventually we choose  $\varepsilon = \mathcal{O}(N)$ , where  $N$  is a large integer, so that  $0 < \varepsilon_0 \leq \varepsilon N \ll 1$ .

The so called time function  $f = \frac{t}{3} + \langle \xi \rangle^{-2/3}$  plays an important role in the calculus of pseudodifferential operators with order function  $m_N^t = f^{-N}(t, \xi)$  and metric

$$g_{(x, \xi)}^\varepsilon = \varepsilon^2 |dx|^2 + \langle \xi \rangle^{-2} |d\xi|^2.$$

The issue of this choice is that the commutator  $[a_2^\varepsilon(t, x, D_x), f^{-N}(t, D_x)]$  has symbol in the space  $S(f^{-N}\varepsilon N \langle \xi \rangle, g^\varepsilon)$  so  $[a_2^\varepsilon(t, x, D_x), f^{-N}(t, D_x)]f^N(t, D_x)$  becomes a first order operator whose norm is not depending on  $N$  and, in particular, on  $\Pi$ . This is proved in Proposition 5.2.

The main idea is to multiply  $\mathcal{P}u$  by the *multiplier*

$$Mu = D_t^2 - \theta t a_2^\varepsilon(t, x, D_x)u,$$

where we choose  $\theta = 1/3$ , and to study the expression

$$-2 \operatorname{Im} \langle f^{-2N}(t, D_x) \mathcal{P}u, Mu \rangle, \quad (1.8)$$

$\langle, \rangle$  denoting the scalar product in  $L^2(\mathbb{R}^n)$  and  $u \in C_0^\infty([0, T] \times U)$ . The above expression modulo lower order terms is a sum  $\sum_{j=1}^{10} I_j$  of ten terms and we make a quite detailed analysis of all these terms in Section 5. The purpose is to find, by integration by parts, “positive” terms with a big coefficient of order  $\mathcal{O}(N)$  which will absorb in the energy estimate the contributions with “indefinite” (possibly negative) sign.

In fact, we have many indefinite terms, while the positive ones come from the expression

$$\partial_t \mathcal{E}_N(u) + 2N \mathcal{E}_{N+1/2}(u),$$

where

$$\begin{aligned} \mathcal{E}_N(u) &= \|f^{-N}u''\|_0^2 + (1 - \theta)t \operatorname{Re} \langle f^{-2N}a_2^\varepsilon u', u' \rangle + \theta t^2 \|f^{-N}a_2^\varepsilon u\|_0^2 \\ &\quad + \theta t^2 \operatorname{Re} \langle f^{-2N}a_2^\varepsilon u, u'' \rangle \\ &\quad + \varepsilon^{1/3} t^2 2 \operatorname{Im} \langle f^{-2N}a_3^\varepsilon u, u' \rangle + \varepsilon^{1/3} \theta t^2 2 \operatorname{Im} \langle f^{-2N}a_1^\varepsilon a_2^\varepsilon u, u' \rangle. \end{aligned}$$

The quantities in the third line above, involving powers of  $\varepsilon$  and  $t$ , are lower order perturbations. We introduce the **energy** of order  $k \in N$  by the expression

$$E_k(u) = \frac{1}{3} \left( \|f^{-k}u''\|_0^2 + 2t \operatorname{Re} \langle f^{-k}a_2^\varepsilon u', f^{-k}u' \rangle + \frac{1}{2}t^2 \|f^{-k}a_2^\varepsilon u\|_0^2 \right), \quad (1.9)$$

where  $a_2^\varepsilon$  is the operator with symbol  $a_2^\varepsilon(0, x, \xi)$  and  $\|\cdot\|_k$  denotes the  $H^k(\mathbb{R}^n)$  norms. Next we prove that modulo lower order terms we have

$$\mathcal{E}_k(u) \geq E_k(u).$$

To gain control on the terms involving  $\|f^{-k}u\|_j$  norms with  $j = 1, 2$  and  $k = N, N + 1/2$ , we would like to exploit the terms with  $H^2(\mathbb{R}^n)$  norms having large coefficients. However, the latter terms have factors  $t$  or  $t^2$  so that  $tN$  or  $t^2N$  becomes close to 0. A way around this is to use the key inequality

$$\langle \xi \rangle^2 f^{-k} \leq t \langle \xi \rangle^2 f^{-k-1} + f^{k-3},$$

established in Lemma 6.1. Thus, for example, we have an estimate

$$\|f^{-N}u\|_1^2 \leq t \|f^{N-1/2}u\|_1^2 + \|f^{-N-3/2}u\|_0^2.$$

For  $H^2$  norms we must apply the above equality twice to gain  $t^2$  factors and this leads to norms of the form  $\|f^{-N-5/2}u\|_0$ . Following this way, we have negative terms with coefficients  $t$  and  $t^2$ , but we have now generated other negative terms involving norms with *weights*  $f^{-N-3/2}$  and  $f^{-N-5/2}$  and these also have to be absorbed. Now, the latter types of terms are not included in our **energy** expression (1.9). To be able to treat such terms, we use the well known spectral shift trick, multiplying our basic inequality by  $e^{-\lambda t}$ ,  $\lambda$  being a large positive parameter. Next we apply

to just a fraction of the positive terms in  $e^{-\lambda t}E_N(u)$ , having a large coefficient proportional to  $N$ , the inequality

$$\begin{aligned} e^{-\lambda t}\|f^{-N-\frac{1}{2}}\partial_t^2 u\|^2 &\geq \partial_t \left( e^{-\lambda t}\|f^{-N-1}\partial_t u\|_0^2 \right) + \lambda e^{-\lambda t}\|f^{-N-1}\partial_t u\|_0^2 \\ &\quad + 2Ne^{-\lambda t}\|f^{-N-\frac{3}{2}}\partial_t u\|_0^2 \\ &\quad + \partial_t \left( e^{-\lambda t}\|f^{-N-2}u\|_0^2 \right) + \lambda e^{-\lambda t}\|f^{-N-2}u\|_0^2 \\ &\quad + (2N+3)e^{-\lambda t}\|f^{-N-\frac{5}{2}}u\|_0^2. \end{aligned} \quad (1.10)$$

A similar inequality for  $e^{-\lambda t}\|f^{-N-1}\partial_t u\|_0^2$  completes our technical toolkit.

Finally, in Section 7 we show that we can absorb all non positive terms and in Section 8 we get the energy estimates with loss of  $2N/3 - 2$  derivatives in  $x$  which imply, by a standard argument, the well posedness of the Cauchy problem.

## 2. HYPERBOLIC OPERATORS WITH TRIPLE CHARACTERISTICS ON $t = 0$

In this section we use the notations of Section 1. According to the hypothesis  $(H_1)$ ,  $P$  has triple characteristics for  $t = 0$  and the triple root of (1.3) for  $t = 0, x \in \bar{U}, \xi \in \mathbb{R}^n$  is

$$\tau = -\frac{1}{3}q_1(0, x, \xi).$$

Let  $-\frac{1}{3}q_1(t, x, \xi) = \sum_{j=1}^n \alpha_j(t, x)\xi_j$ . We will show that after a change of variables

$$s = t, y_j = f_j(t, x), j = 1, \dots, n,$$

this triple root for  $s = 0$  becomes  $\sigma = 0$ , where we denote by  $(\sigma, \eta)$  the variables dual to  $(s, y)$ . We have

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial}{\partial s} + \sum_{j=1}^n \frac{\partial f_j}{\partial t} \frac{\partial}{\partial y_j}, \\ \frac{\partial}{\partial x_k} &= \sum_{j=1}^n \frac{\partial f_j}{\partial x_k} \frac{\partial}{\partial y_j}, \quad k = 1, \dots, n. \end{aligned}$$

Then

$$\begin{aligned} &\frac{\partial}{\partial t} - \sum_{j=1}^n \alpha_j(t, x) \frac{\partial}{\partial x_j} \\ &= \frac{\partial}{\partial s} + \sum_{j=1}^n \left( \frac{\partial f_j}{\partial t} - \sum_{k=1}^n \alpha_k(t, x) \frac{\partial f_j}{\partial x_k} \right) \frac{\partial}{\partial y_j}. \end{aligned}$$

Now we choose

$$f_j(t, x) = x_j + t\alpha_j(0, x), \quad j = 1, \dots, n$$

and we conclude that for  $t = 0$  the operator  $\frac{\partial}{\partial t} - \sum_{j=1}^n \alpha_j(0, x) \frac{\partial}{\partial x_j}$  is transformed in  $\frac{\partial}{\partial s}$ . Moreover, for  $t = 0$  for the Jacobian we get

$$\frac{D(s, y)}{D(t, x)} = I.$$

Since  $\bar{U}$  is compact and the coefficients  $\alpha_j(0, x)$  are bounded for  $x \in \bar{U}$ , we conclude that the above change is non degenerate for  $t \geq 0$  small enough and  $x \in \bar{U}$ .



When, for  $t = 0$  and  $(x, \xi)$  in a conic neighborhood of  $(x_0, \xi_0) \in T^*(U) \setminus \{0\}$ , we have a triple root  $\tau = -\frac{1}{3}q_1(0, x, \xi) = \lambda(x, \xi)$  of the equation  $p_3 = 0$ , we may find a homogeneous canonical transformation, being just a change of variables in the differential case,

$$s = t, \quad y_j = Y_j(t, x, \xi), \quad \sigma = \tau - \lambda(x, \xi), \quad \eta_j = Z_j(t, x, \xi), \quad j = 1, \dots, n,$$

defined in a small conic neighborhood of  $(0, x_0, \lambda(x_0, \xi_0), \xi_0)$  in  $T^*(\mathbb{R} \times U) \setminus \{0\}$  so that  $\tau - \lambda(x, \xi) = 0$  is transformed into  $\sigma = 0$ . For this purpose it is sufficient to observe that the Poisson bracket  $\{\tau - \lambda(x, \xi), t\} = 1$  and the three one-forms

$$d\tau - d\lambda(x, \xi), \quad dt, \quad \tau dt + \sum_{j=1}^n \xi_j dx_j$$

are linearly independent. Then the results follows from Theorem 21.1.9 in [6]. Consequently, microlocally we can reduce the analysis to the case examined in our paper.

In the new variables which we denote again by  $(t, x)$  the operator  $P$  is transformed into

$$\begin{aligned} \mathcal{P} = & D_t^3 + ta_1(t, x, D_x)D_t^2 - ta_2(t, x, D_x)D_t + t\tilde{a}_3(t, x, D_x) + b_2(t, x, D_x) \\ & + b_1(t, x, D_x)D_t + b_0(t, x)D_t^2 + c_1(t, x, D_x) + c_0(t, x)D_t + d_0(t, x). \end{aligned} \quad (2.1)$$

Here  $a_1, a_2, \tilde{a}_3$  are homogeneous polynomials with respect to  $\xi$  respectively of order 1, 2, 3,  $b_j(t, x, \xi)$  are homogeneous polynomials with respect to  $\xi$  of order  $j$ , while  $b_0, c_0, d_0$  are smooth functions and  $c_1(t, x, \xi)$  is a first order differential operator with respect to the variable  $x$ .

In the following throughout our exposition we will assume that the principal symbol  $p_3(0, x, \tau, \xi)$  of  $\mathcal{P}$  has a triple root  $\tau = 0$  for  $t = 0, x \in \bar{U}, \xi \in \mathbb{R}^n$  and we examine the operator having the form (2.1). It is clear that the Hamilton map of  $p_3$  for  $t = \tau = 0$  has non-vanishing eigenvalues if only if  $a_2(0, x, \xi) \neq 0, x \in \bar{U}, \xi \in \mathbb{R}^N \setminus 0$ .

Notice that for  $|\xi| = 1$  the discriminant  $\Delta$  of the equation  $p_3(t, x, \tau, \xi) = 0$  with respect to  $\tau$  has the form

$$\begin{aligned} \Delta(t, x, \xi) &= \left( \frac{-3ta_2 - t^2a_1^2}{9} \right)^3 + \left( \frac{9t^2a_1a_2 + 27t\tilde{a}_3 + 2t^3a_1^3}{54} \right)^2 \\ &= Q^3 + R^2 = \frac{1}{4}t^2\tilde{a}_3^2 - \frac{1}{27}t^3a_2^3 + \frac{1}{6}t^3a_1a_2\tilde{a}_3 + \mathcal{O}(t^4)a_6. \end{aligned}$$

If at a point  $(x_0, \xi_0) \in T^*(U) \setminus \{0\}$  we have  $\tilde{a}_3(0, x_0, \xi_0) \neq 0$ , then for small  $t > 0$  we get  $\Delta(t, x_0, \xi_0) > 0$  and the cubic equation  $p_3 = 0$  has complex roots. Thus  $\tilde{a}_3(0, x, \xi) = 0$  for  $(x, \xi) \in T^*(U)$ . By the same argument (see also Lemma 8.1 in [7]) we conclude that  $a_2(0, x, \xi) > 0, \forall (x, \xi) \in T^*(\bar{U}) \setminus \{0\}$ . Consequently,  $\Delta < 0$  for small  $t > 0$  and  $x \in \bar{U}, |\xi| = 1$ . Thus the operator  $\mathcal{P}$  is strictly hyperbolic for small  $t > 0$  and to study the Cauchy problem for  $0 \leq t \leq T$  with sufficiently small  $T > 0$ , it suffices to examine the Cauchy problem for  $0 \leq t \leq t_0 < T, t_0 \ll 1$ .

Now we change the notations and write  $\tilde{a}_3(t, x, \xi) = ta_3(t, x, \xi)$ . Then the principal part of  $\mathcal{P}$  has the form

$$\mathcal{P}_3 = D_t^3 + ta_1(t, x, D_x)D_t^2 - ta_2(t, x, D_x)D_t + t^2a_3(t, x, D_x)$$

with  $a_j(t, x, \xi), j = 1, 2, 3$  real-valued polynomials of order  $j$  in  $\xi$  and

$$a_2(0, x, \xi) \geq c|\xi|^2, \quad c > 0 \text{ for } x \in \bar{U}, \xi \neq 0.$$

It is well known that the subprincipal symbol and the eigenvalues of the Hamilton map are invariant on the characteristic points  $\rho \in \Sigma_1 = \{\rho \in T^*(G) \setminus \{0\} : p_3(\rho) = 0, dp_3(\rho) = 0\}$ . Thus the number  $\Pi$  defined by (1.6) can be expressed by the subprincipal symbol  $p'_2(0, x, \xi)$  and  $a_2(0, x, \xi)$ .

We extend the coefficients of  $a_j, j = 1, 2, 3, b_k, k = 0, 1, 2$  and  $c_1, c_0, d_0$  for  $x \in \mathbb{R}^n$  as smooth functions. Thus in the analysis in Section 3-8 we will assume that the operator  $\mathcal{P}$  are defined in  $\mathbb{R}^n$ . Moreover, our arguments works without changes if  $a_j, b_k, c_k$ , etc, are classical pseudodifferential operators with symbols

$$a_j(t, x, \xi) \in S_{1,0}^j(\mathbb{R}^{n+1} \times \mathbb{R}^n), b_k(t, x, \xi) \in S_{1,0}^k(\mathbb{R}^{n+1} \times \mathbb{R}^n), c_k(t, x, \xi) \in S_{1,0}^k(\mathbb{R}^{n+1} \times \mathbb{R}^n)$$

depending smoothly on the parameter  $t$ .

We can obtain more information for the real roots of  $p_3(t, x, \tau, \xi)$  for small  $t$ . Notice that the coefficients of the cubic equation  $p_3(t, x, \tau, \xi) = 0$  are real, and for  $t \geq 0$  its real roots  $\lambda_k(t, x, \xi)$ ,  $k = 1, 2, 3$ , have the following trigonometric form (see for instance, [16])

$$\begin{cases} \lambda_1 = 2\rho^{1/3} \cos(\theta/3) - \frac{ta_1}{3}, \\ \lambda_2 = 2\rho^{1/3} \cos(\theta/3 + \frac{2\pi}{3}) - \frac{ta_1}{3}, \\ \lambda_3 = 2\rho^{1/3} \cos(\theta/3 + \frac{4\pi}{3}) - \frac{ta_1}{3}, \end{cases}$$

where

$$\rho = (-Q)^{3/2}, \quad \theta = \arccos(R/\rho)$$

with  $Q = -\frac{3ta_2 + t^2a_1^2}{9}$ . Next consider the symbols

$$\delta_k = \frac{\partial p_3}{\partial \tau} \Big|_{\tau=\lambda_k} = \left( 3\tau^2 + 2ta_1\tau - ta_2 \right) \Big|_{\tau=\lambda_k}, \quad k = 1, 2, 3.$$

Since these symbols are homogeneous of order 2 in  $\xi$ , to find lower bounds for  $|\delta_k|$ , it is sufficient to examine their behavior for  $|\xi| = 1$ . We have

$$\delta_1 = 12\rho^{2/3} \cos^2(\theta/3) - ta_2 + \mathcal{O}(t^{3/2})a_2 = \left( 4\cos^2(\theta/3) - 1 \right) ta_2 + \mathcal{O}(t^{3/2})a_2.$$

On the other hand,  $\frac{t}{\rho} = \mathcal{O}(t^{1/2})$ , hence  $\cos(\theta/3) = \frac{\sqrt{3}}{2} + o(t)$  and this implies for small  $t$  and  $x \in \bar{U}$ ,  $|\xi| = 1$  the estimate  $|\delta_1| \geq c_1 ta_2$  with  $c_1 > 0$ . Moreover,

$$\delta_{2,3} = 3\lambda_{2,3}^2 - ta_2 + \mathcal{O}(t^{3/2})a_2 = \left( 4\sin^2(\pi/6 \pm \theta/3) - 1 \right) ta_2 + \mathcal{O}(t^{3/2})a_2$$

and we obtain the following

**Lemma 2.1.** *There exist constants  $\gamma > 0$  and  $\gamma_1 > 0$  such that for  $0 \leq t \leq \gamma_1, x \in \bar{U}$  we have*

$$|\delta_k| \geq \gamma ta_2(t, x, \xi) \geq \gamma ct|\xi|^2, \quad k = 1, 2, 3. \quad (2.2)$$

In our exposition we will not use Lemma 2.1 which has an independent interest.

### 3. SOME CLASSES OF SYMBOLS

Let

$$f(t, \xi) = \frac{t}{3} + \langle \xi \rangle^{-2/3}, \quad (3.1)$$

where  $\langle \xi \rangle^2 = 1 + |\xi|^2$ . Then clearly  $f$  is a symbol in the class  $S_{1,2/3}^0$ , when derivatives with respect to  $t$  are considered, but it is in the class  $S_{1,0}^0$  if  $t$  is just a parameter and no derivatives with respect to  $t$  are involved.

It will be convenient for us to use the Weyl calculus formalism, in the variables  $x$ , in order to establish an *a priori* estimate for the operator we deal with. From now on  $t$  will be regarded as a non-negative parameter.

Let  $\varepsilon > 0$  be a small positive number. We consider the metric in  $T^*\mathbb{R}^n$  defined by

$$g_{(x,\xi)}^\varepsilon = \varepsilon^2 |dx|^2 + \langle \xi \rangle^{-2} |d\xi|^2 \quad (3.2)$$

which is almost the classical  $(1,0)$ -metric. In the following we will write  $g$ , when there is no ambiguity. It is well known that  $g$  is a slowly varying metric.

Let  $N$  be a positive integer. In what follows the size of  $N$  is determined in terms of the problem. We define the "order" function

$$m_N^t(\xi) = f^{-N}(t, \xi). \quad (3.3)$$

It is trivial to verify that  $m_N^t$  is an order function. Then we may define the classes  $S(m_N^t, g^\varepsilon)$  of symbols in the standard way.

We point out explicitly that  $t$  is just a parameter at this level and that if there is no ambiguity we may omit it in our notation. We have the following

**Proposition 3.1.**  $f^{-N}(t, \xi) \in S(m_N^t, g^\varepsilon)$ .

*Proof.* Of course we must check only  $\xi$ -derivatives. We have that

$$\partial_{\xi_j} f^{-N}(t, \xi) = -N f^{-N}(t, \xi) \frac{\langle \xi \rangle^{-2/3}}{f(t, \xi)} \left( -\frac{2}{3} \right) \frac{\xi_j}{\langle \xi \rangle} \langle \xi \rangle^{-1}.$$

Hence we have the estimate  $|\partial_{\xi_j} f^{-N}(t, \xi)| \leq C_N f^{-N}(t, \xi) \langle \xi \rangle^{-1}$ . A simple iteration concludes the proof.  $\square$

**Remark 3.1.** In particular we deduce that

$$\partial_\xi^\alpha f^{-N}(t, \xi) = \mathcal{O} \left( N^{|\alpha|} f^{-N}(t, \xi) \langle \xi \rangle^{-|\alpha|} \right).$$

Given a symbol  $a(t, x, \xi) \in S(m_N^t, g^\varepsilon)$ , which we may also denote by  $a^t(x, \xi)$ , the Weyl pseudo-differential operator associated with it is defined by the formula

$$a^{tw}u(x) = (2\pi)^{-n} \int \int e^{i\langle x-y, \xi \rangle} a^t \left( \frac{x+y}{2}, \xi \right) u(y) dy \, d\xi.$$

We recall the composition rule for two symbols (see e.g. [6]). Define

$$g^\sigma(w) = \sup_{w'} \frac{|\sigma(w, w')|^2}{g(w')}.$$

Here  $\sigma$  denotes the symplectic form defined on  $T(T^*\mathbb{R}^n) \times T(T^*\mathbb{R}^n)$ , which in our local coordinates is given by

$$\sigma((x, \xi), (y, \eta)) = \langle y, \xi \rangle - \langle x, \eta \rangle.$$

**Theorem 3.1.** *Let  $g$  be a temperate metric with  $g \leq g^\sigma$  and let  $m_1, m_2$  be  $(\sigma, g)$  temperate order functions. Let  $a_j \in S(m_j, g)$ ,  $j = 1, 2$ . Then the composition of the associated pseudodifferential operators is associated to a symbol map  $(a_1, a_2) \mapsto a = a_1 \# a_2$  from  $S(m_1, g) \times S(m_2, g)$  to  $S(m_1 m_2, g)$  and  $a$  is defined by*

$$a(x, \xi) = \exp \left( \frac{i}{2} \sigma(D_x, D_\xi; D_y, D_\eta) \right) a_1(x, \xi) a_2(y, \eta) \Big|_{(x, \xi) = (y, \eta)}. \quad (3.4)$$

Let

$$h(x, \xi)^2 = \sup_w \frac{g_{x, \xi}(w)}{g_{x, \xi}^\sigma(w)}, \quad (3.5)$$

then we have that for every integer  $M$  the map associating  $a_1, a_2$  to the remainder term

$$a_1 \# a_2(x, \xi) - \sum_{j < M} \frac{(i \sigma(D_x, D_\xi; D_y, D_\eta))^j}{2^j j!} a_1(x, \xi) a_2(y, \eta)$$

evaluated on the diagonal  $(x, \xi) = (y, \eta)$ , is continuous with values in  $S(h^M m_1 m_2, g)$ .

**Remark 3.2.** *We explicitly note that the above formula (3.4) reduces to the usual symbol composition formula (i.e. with no effect of the Weyl operator definition) if  $a_1$  or  $a_2$  does not depend on  $x$ ; thus (3.4) reduces to*

$$a(x, \xi) = \sum_{|\alpha| \geq 0} \frac{1}{\alpha!} \partial_\xi^\alpha a_1(\xi) D_x^\alpha a_2(x, \xi),$$

or the analogous symmetric formula if  $a_2$  is independent of  $x$ .

Note that a different way of writing the above formula is

$$a(x, \xi) = \exp \left( \frac{i}{2} \langle D_y, D_\xi \rangle \right) a_1(\xi) a_2(y, \eta) \Big|_{(x, \xi) = (y, \eta)}.$$

**Remark 3.3.** *An easy and explicit calculation yields that*

$$(g_{x, \xi}^\varepsilon)^\sigma = \langle \xi \rangle^2 |dx|^2 + \varepsilon^{-2} |d\xi|^2, \quad (3.6)$$

and consequently the function  $h$  is given by

$$h(x, \xi) = \frac{\varepsilon}{\langle \xi \rangle}. \quad (3.7)$$

Evidently in this case

$$g_{x, \xi}^\varepsilon \leq (g_{x, \xi}^\varepsilon)^\sigma.$$

We need to have a better control of this remainder terms in order to estimate the composition symbol with respect to the parameter  $N$  introduced above.

We recall a result due to J.-M. Bony, [2], according to which the composition  $a_1 \# a_2$  is written as a finite sum plus a remainder term:

$$a_1 \# a_2(x, \xi) = \sum_{p=0}^{M-1} \frac{1}{p!} \left( \frac{i}{2} \sigma(D_x, D_\xi; D_y, D_\eta) \right)^p a_1(x, \xi) a_2(y, \eta) \Big|_{(x, \xi) = (y, \eta)} + R_M(a_1, a_2)(x, \xi), \quad (3.8)$$

where

$$R_M(a_1, a_2)(x, \xi) = \int_0^1 \frac{(1-\theta)^{M-1}}{(M-1)!} \cdot \frac{1}{(\pi\theta)^{2n}} \int \int e^{-(2i/\theta)\sigma((x,\xi)-(t,\tau), (x,\xi)-(y,\eta))} \cdot \left( \frac{i}{2} \sigma(D_t, D_\tau; D_y, D_\eta) \right)^M a_1(t, \tau) a_2(y, \eta) dt d\tau dy d\eta d\theta \quad (3.9)$$

**Proposition 3.2.** *Assume that  $a_i \in S(m_i, g)$ , where  $g$  is a slowly varying, temperate metric such that  $g \leq g^\sigma$ . Then both  $R_M$  and the restriction to the diagonal of*

$$\sigma((D_x, D_\xi); (D_y, D_\eta)) a_1(x, \xi) a_2(y, \eta)$$

*belong to  $S(m_1 m_2 h^M, g)$ .*

#### 4. SCALING AND MULTIPLIER

To obtain an a priori estimate and to deal with the lower order terms we introduce a scaling

$$t = \varepsilon^{2/3} s, \quad x = \varepsilon y, \quad \varepsilon > 0. \quad (4.1)$$

Multiplying by  $\varepsilon^2$ , we obtain an operator

$$\begin{aligned} \mathcal{P} = & D_s^3 - s a_2(\varepsilon^{2/3} s, \varepsilon y, D_y) D_s + b_2(\varepsilon^{2/3} s, \varepsilon y, D_y) \\ & + \varepsilon^{1/3} \left[ s a_1(\varepsilon^{2/3} s, \varepsilon y, D_y) D_s^2 + s^2 a_3(\varepsilon^{2/3} s, \varepsilon y, D_y) + b_1(\varepsilon^{2/3} s, \varepsilon y, D_y) D_s \right] \\ & + \varepsilon^{2/3} b_0(\varepsilon^{2/3} s, \varepsilon y) D_s^2 + \varepsilon c_1(\varepsilon^{2/3} s, \varepsilon y, D_y) + \varepsilon^{4/3} c_0(\varepsilon^{2/3} s, \varepsilon y) D_s + \varepsilon^2 d_0(\varepsilon^{2/3} s, \varepsilon y). \end{aligned} \quad (4.2)$$

Here we use for simplicity the same notation  $\mathcal{P}$  for the transformed operator. Moreover,  $a_j, j = 1, 2, 3, b_k, k = 0, 1, 2$ , etc. are the symbols of Section 2.

Since we are interested in obtaining an estimate for  $0 \leq t \leq T$  with sufficiently small  $T > 0$ , we may think of  $\varepsilon$  as a parameter which is going to be chosen sufficiently small; actually it will be fixed below as  $\varepsilon = \mathcal{O}(\frac{1}{N})$ , where  $N = 3\sqrt{3}\Pi + N_0$  and  $\Pi$  was defined in the Introduction.

Let

$$P_0 = D_s^3 - s a_2(0, \varepsilon y, D_y) D_s + b_2(0, \varepsilon y, D_y) \quad (4.3)$$

be the leading term in  $\mathcal{P}$  having no factors depending on  $\varepsilon$ .

It is convenient to use the following notation:

$$a_j^\varepsilon(t, x, D_x) = a_j(\varepsilon^{2/3} t, \varepsilon x, D_x), \quad j = 1, 2, 3 \quad (4.4)$$

and

$$b_2^\varepsilon(t, x, D_x) = b_2(\varepsilon^{2/3} t, \varepsilon x, D_x), \quad (4.5)$$

for the second order differential operators appearing in the definition of  $\mathcal{P}$  (4.2), emphasizing the dependence on the parameter  $\varepsilon$ . We may also return to the notation  $(t, x)$  for the time and space variables respectively, without any risk of misunderstanding.

For  $u, v \in C_0^\infty(\overline{\mathbb{R}^+} \times \mathbb{R}^n)$ , we denote by

$$\langle u, v \rangle = \int_{\Omega} u(t, x) \bar{v}(t, x) dx,$$

the usual scalar product in  $L^2(\mathbb{R}^n)$  w.r.t. the space variables  $x$ . Also we denote by  $\|\cdot\|_k$  the norms in the spaces  $H^k(\mathbb{R}^n)$ .

In order to deduce an energy estimate, we need a second order multiplier operator. In what follows we use the multiplier

$$M(t, x, D_t, D_x) = D_t^2 - \theta t a_2^\varepsilon(t, x, D_x), \quad (4.6)$$

where  $a_2^\varepsilon$  is the same operator appearing in the definition of  $\mathcal{P}$  and  $\theta$  denotes a positive constant to be chosen later.

Let us compute, for  $u \in C_0^\infty(\overline{\mathbb{R}^+} \times \mathbb{R}^n)$ , the expression

$$-2 \operatorname{Im} \langle f^{-2N}(t, D_x) \mathcal{P}u, Mu \rangle.$$

Here  $f^{-2N}(t, D_x)$  denotes the pseudodifferential operator whose symbol is  $f^{-2N}(t, \xi) \in S(m_{2N}^t, g^\varepsilon)$ .

We have

$$\begin{aligned} -2 \operatorname{Im} \langle f^{-2N}(t, D_x) \mathcal{P}u, Mu \rangle &= 2 \operatorname{Re} \langle f^{-2N} (\partial_t^3 + t a_2^\varepsilon \partial_t) u, (\partial_t^2 + \theta t a_2^\varepsilon) u \rangle \\ &\quad - 2 \operatorname{Im} t \langle f^{-2N} \varepsilon^{1/3} a_1^\varepsilon(t, x, D_x) \partial_t^2 u, (\partial_t^2 + \theta t a_2^\varepsilon) u \rangle \\ &\quad + 2 \operatorname{Im} \langle f^{-2N} \varepsilon^{1/3} t^2 a_3^\varepsilon(t, x, D_x) u, (\partial_t^2 + \theta t a_2^\varepsilon) u \rangle \\ &\quad + 2 \operatorname{Im} \langle f^{-2N} b_2^\varepsilon(t, x, D_x) u, (\partial_t^2 + \theta t a_2^\varepsilon) u \rangle \\ &\quad + \text{lower order terms.} \end{aligned} \quad (4.7)$$

Here we denoted by "lower order terms" the terms of order 1 or 2 involving the operators  $b_1, d_0, d_1$ . It will be evident after the discussion below that they do not have any influence whatsoever on the energy estimate for  $\mathcal{P}$  that we are going to deduce and hence, to avoid burdening the exposition with useless details we omit a discussion of those terms.

Next we list in a more detailed way the terms in (4.7) above.

$$\begin{aligned} -2 \operatorname{Im} \langle f^{-2N}(t, D_x) \mathcal{P}u, Mu \rangle &= 2 \operatorname{Re} \langle f^{-2N} \partial_t^3 u, \partial_t^2 u \rangle + 2 \operatorname{Re} \langle f^{-2N} \partial_t^3 u, \theta t a_2^\varepsilon u \rangle \\ &\quad + 2 \operatorname{Re} \langle f^{-2N} t a_2^\varepsilon \partial_t u, \partial_t^2 u \rangle + 2 \operatorname{Re} \langle f^{-2N} t a_2^\varepsilon \partial_t u, \theta t a_2^\varepsilon u \rangle \\ &\quad - 2 \operatorname{Im} \langle f^{-2N} \varepsilon^{1/3} t a_1^\varepsilon \partial_t^2 u, \partial_t^2 u \rangle - 2 \operatorname{Im} \langle f^{-2N} \varepsilon^{1/3} t a_1^\varepsilon \partial_t^2 u, \theta t a_2^\varepsilon u \rangle \\ &\quad + 2 \operatorname{Im} \langle f^{-2N} \varepsilon^{1/3} t^2 a_3^\varepsilon u, \partial_t^2 u \rangle + 2 \operatorname{Im} \langle f^{-2N} \varepsilon^{1/3} t^2 a_3^\varepsilon u, \theta t a_2^\varepsilon u \rangle \\ &\quad + 2 \operatorname{Im} \langle f^{-2N} b_2^\varepsilon(t, x, D_x) u, \partial_t^2 u \rangle + 2 \operatorname{Im} \langle f^{-2N} b_2^\varepsilon(t, x, D_x) u, \theta t a_2^\varepsilon u \rangle \\ &\quad + \text{lower order terms} \\ &= \sum_{j=1}^{10} I_j + \text{lower order terms} \end{aligned} \quad (4.8)$$

In the next section we are going to estimate each term  $I_j$  with the purpose of putting in evidence a positive energy containing the weight  $f^{-N}$  as well as  $f^{-N-1/2}$ .

## 5. ESTIMATE OF THE TERMS IN (4.8)

**5.1. Estimate of  $I_1$ .** For the first term we have

$$I_1 = 2 \operatorname{Re} \langle f^{-2N} \partial_t^3 u, \partial_t^2 u \rangle = \partial_t \|f^{-N} \partial_t^2 u\|^2 + 2N \|f^{-N-1/2} \partial_t^2 u\|^2. \quad (5.1)$$

We write, here as well as in the preceding section,  $f^{-2N}$  instead of  $\text{Op}(f^{-2N})$ , for the sake of simplicity.

Note that we have used the fact that  $f$ , or rather its powers, is self adjoint as an operator w.r.t. the  $x$  variables when acting on smooth functions with compact support.

**5.2. Estimate of  $I_3$ .** Due to Proposition 3.1, we have that, as a symbol,  $f^{-2N} \in S(m_{2N}^t, g^\varepsilon)$ , where  $\varepsilon$  is a positive parameter to be chosen below and the variable  $t$  can be regarded as a parameter for the time being. The order function  $m_{2N}^t$  has been defined in (3.3).

Taking into account that we performed a dilation by  $\varepsilon$ , we conclude that

**Proposition 5.1.** *Both symbols  $a_2^\varepsilon$  and  $b_2^\varepsilon$  belong to  $S(\langle \xi \rangle^2, g^\varepsilon)$ , as symbols in the  $x$  variables. It is then straightforward to show that actually*

$$\varepsilon^{-\frac{2}{3}j} \partial_t^j a(t, x, \xi) \in S(\langle \xi \rangle^2, g^\varepsilon), \quad (5.2)$$

where  $a$  denotes either  $a_2^\varepsilon$  or  $b_2^\varepsilon$ .

A typical situation we encounter in the estimate of  $I_j$  is the evaluation of a norm or scalar product involving a commutator. We have

**Proposition 5.2.** *The commutator*

$$[a_2^\varepsilon(t, x, D_x), f^{-2N}] \quad (5.3)$$

has a symbol in  $S(f^{-2N} N \varepsilon \langle \xi \rangle, g^\varepsilon)$ .

**Corollary 5.1.** *If  $0 < \varepsilon \leq \varepsilon_0$ , where  $\varepsilon_0$  denotes a suitably small positive number depending on  $N$ , then the commutator in (5.3) can be written as*

$$[a_2^\varepsilon(t, x, D_x), f^{-2N}] = f^{-2N} \gamma_1^\varepsilon(t, x, D_x), \quad (5.4)$$

where  $\gamma_1^\varepsilon \in S(\langle \xi \rangle, g^\varepsilon)$ .

*Proof of Proposition 5.2.* Since  $f^{-2N}$  does not depend on  $x$ , the bracket can be written as a product:

$$\text{symb}([a_2^\varepsilon(t, x, D_x), f^{-2N}(t, D_x)]) = a_2^\varepsilon f^{-2N} - f^{-2N} \# a_2^\varepsilon,$$

where  $\text{symb}(b)$  denotes the symbol of the operator  $b$ .

Using Proposition 3.2, as well as definitions (3.2) and (3.5), we obtain that the r.h.s. of the above identity belongs to  $S(f^{-2N} \varepsilon \langle \xi \rangle, g^\varepsilon)$ .  $\square$

*Proof of Corollary 5.1.* Choosing  $M = 1$  in (3.9) we get

$$\begin{aligned} & \sigma([a_2^\varepsilon(t, x, D_x), f^{-2N}(t, D_x)])(t, x, \xi) \\ &= \int_0^1 \frac{1}{(\pi\theta)^{2n}} \int \int e^{-(2i/\theta)\sigma((x,\xi)-(z,\zeta), (x,\xi)-(y,\eta))} \frac{i}{2} D_z a_2^\varepsilon(t, z, \zeta) \\ & \quad \cdot D_\eta f^{-2N}(t, \eta) dz d\zeta dy d\eta d\theta. \end{aligned}$$

Since

$$\partial_z a_2^\varepsilon \partial_\eta f^{-2N} = \frac{4N}{3} \varepsilon \langle (\partial_z a_2)^\varepsilon, \frac{\eta}{\langle \eta \rangle} \rangle f^{-2N} \langle \eta \rangle^{-1} \frac{\langle \eta \rangle^{-2/3}}{t + \langle \eta \rangle^{-2/3}},$$

we see that besides the order function  $f^{-2N} \varepsilon \langle \xi \rangle$  we have also a factor  $N$ , which justifies the presence of  $\varepsilon$ . Here we used the notation  $(\partial_z a_2)^\varepsilon$  to denote the symbol  $(\partial_z a_2)(\varepsilon^{2/3}t, \varepsilon z, \xi)$ . See also definition (4.4).  $\square$

Due to the above statements we may conclude that

$$[f^{-2N}, a_2^\varepsilon] = f^{-2N} \alpha_1^\varepsilon, \quad (5.5)$$

for some first order symbol  $\alpha_1^\varepsilon$ . Therefore

$$\begin{aligned} I_3 &= t (\langle f^{-2N} a_2^\varepsilon u', u'' \rangle + \langle u'', (a_2^\varepsilon f^{-2N} + f^{-2N} \alpha_1^\varepsilon) u' \rangle) \\ &= t (\langle f^{-2N} a_2^\varepsilon u', u'' \rangle + \langle f^{-2N} a_2^\varepsilon u'', u' \rangle + \langle f^{-2N} u'', \alpha_1^\varepsilon u' \rangle) \\ &= t \partial_t \langle f^{-2N} a_2^\varepsilon u', u' \rangle + 2Nt \langle f^{-2N-1} a_2^\varepsilon u', u' \rangle \\ &\quad - t \langle f^{-2N} \partial_t (a_2^\varepsilon) u', u' \rangle + \langle f^{-2N} u'', \alpha_1^\varepsilon u' \rangle \\ &= t \partial_t \langle f^{-2N} a_2^\varepsilon u', u' \rangle + 2Nt \langle f^{-2N-1} a_2^\varepsilon u', u' \rangle \\ &\quad + I_{3,1} + I_{3,2}. \end{aligned} \quad (5.6)$$

Here we denoted by  $u' = \partial_t u$  and  $u'' = \partial_t^2 u$ . Moreover  $\partial_t (a_2^\varepsilon)$  denotes the operator whose symbol (or coefficients in the differential case) are the  $t$ -derivative of  $a_2^\varepsilon$ .

**5.3. Estimate of  $I_4$ .** Let us consider  $I_4$ . We have

$$\begin{aligned} I_4 &= 2 \operatorname{Re} \langle f^{-2N} t a_2^\varepsilon \partial_t u, \theta t a_2^\varepsilon u \rangle \\ &= \theta t^2 (\langle f^{-2N} a_2^\varepsilon u', a_2^\varepsilon u \rangle + \langle a_2^\varepsilon u, f^{-2N} a_2 u' \rangle) \\ &= \theta t^2 \partial_t \langle f^{-2N} a_2^\varepsilon u, a_2^\varepsilon u \rangle + 2N \theta t^2 \langle f^{-2N-1} a_2^\varepsilon u, a_2^\varepsilon u \rangle \\ &\quad - 2 \theta t^2 \operatorname{Re} \langle f^{-2N} (\partial_t a_2^\varepsilon) u, a_2^\varepsilon u \rangle. \\ &= \theta t^2 \partial_t \langle f^{-2N} a_2^\varepsilon u, a_2^\varepsilon u \rangle + 2N \theta t^2 \langle f^{-2N-1} a_2^\varepsilon u, a_2^\varepsilon u \rangle \\ &\quad + I_{4,1}. \end{aligned} \quad (5.7)$$

Here we just used the fact that both  $f^{-2N}$  and  $a_2^\varepsilon$  are self adjoint in  $L^2(\Omega)$ ,  $t$  being a parameter at this stage.

We point out explicitly that  $I_1$ ,  $I_3$  and  $I_4$  are the main terms concurring to the energy as we shall see in the sequel.

**5.4. Estimate of  $I_5$ .** This is very simple:

$$\begin{aligned} I_5 &= -2 \operatorname{Im} \langle f^{-2N} \varepsilon^{1/3} t a_1^\varepsilon \partial_t^2 u, \partial_t^2 u \rangle \\ &= t \varepsilon^{1/3} 2 \operatorname{Re} \langle f^{-2N} \alpha_0^\varepsilon u'', u'' \rangle, \end{aligned} \quad (5.8)$$

where  $\alpha_0^\varepsilon$  denotes a suitable pseudodifferential operator of order 0. Here we have used the fact that the symbol of  $a_1^\varepsilon(t, x, D_x)$  is real valued, hence  $(a_1^\varepsilon(t, x, D_x))^* = a_1^\varepsilon(t, x, D_x) + \alpha_0^\varepsilon(t, x, D_x)$ .



5.5. **Estimate of  $I_2$ .** Let us consider the expression for  $I_2$ , see (4.8),

$$\begin{aligned}
I_2 &= 2 \operatorname{Re} \langle f^{-2N} \partial_t^3 u, \theta t a_2^\varepsilon u \rangle \\
&= \theta t \left( \langle f^{-2N} u''', a_2^\varepsilon u \rangle + \langle a_2^\varepsilon u, f^{-2N} u''' \rangle \right) \\
&= \theta t \partial_t \left( \langle f^{-2N} u'', a_2^\varepsilon u \rangle + \langle a_2^\varepsilon u, f^{-2N} u'' \rangle - \langle f^{-2N} u', a_2^\varepsilon u' \rangle \right) \\
&\quad + 2N \theta t \left( \langle f^{-2N-1} u'', a_2^\varepsilon u \rangle + \langle a_2^\varepsilon u, f^{-2N-1} u'' \rangle \right. \\
&\quad \left. - \langle f^{-2N-1} u', a_2^\varepsilon u' \rangle \right) \\
&\quad + \theta t 2 \operatorname{Re} \langle f^{-2N} \tilde{\alpha}_1^\varepsilon u', u'' \rangle \\
&\quad - \theta t \left( \langle f^{-2N} u'', (\partial_t a_2^\varepsilon) u \rangle + \langle (\partial_t a_2^\varepsilon) u, f^{-2N} u'' \rangle - \langle f^{-2N} u', (\partial_t a_2^\varepsilon) u' \rangle \right). \\
&= \theta t \partial_t \left( 2 \operatorname{Re} \langle f^{-2N} u'', a_2^\varepsilon u \rangle - \langle f^{-2N} u', a_2^\varepsilon u' \rangle \right) \\
&\quad + 2N \theta t \left( 2 \operatorname{Re} \langle f^{-2N-1} u'', a_2^\varepsilon u \rangle - \langle f^{-2N-1} u', a_2^\varepsilon u' \rangle \right) \\
&\quad + \sum_{k=1}^4 I_{2,k}.
\end{aligned} \tag{5.9}$$

A few words are in order. Here  $\tilde{\alpha}_1^\varepsilon$  denotes a suitable first order pseudodifferential operator originating from a commutator exactly as it occurred for the other terms above.

Moreover in deducing (5.9) the following identity has been used:

$$\begin{aligned}
&\partial_t \left( \langle f^{-2N} u'', a_2^\varepsilon u \rangle + \langle a_2^\varepsilon u, f^{-2N} u'' \rangle - \langle f^{-2N} u', a_2^\varepsilon u' \rangle \right) \\
&= \langle f^{-2N} u''', a_2^\varepsilon u \rangle + \langle f^{-2N} u'', a_2^\varepsilon u' \rangle + \langle a_2^\varepsilon u', f^{-2N} u'' \rangle + \langle a_2^\varepsilon u, f^{-2N} u''' \rangle \\
&\quad - \langle f^{-2N} u'', a_2^\varepsilon u' \rangle - \langle f^{-2N} u', a_2^\varepsilon u'' \rangle + \text{terms being } \mathcal{O}(N) + \text{terms involving } \partial_t a_2^\varepsilon.
\end{aligned}$$

Thus let us examine the first four terms in the r.h.s. above. We have

$$\begin{aligned}
&\langle f^{-2N} u''', a_2^\varepsilon u \rangle + \langle f^{-2N} u'', a_2^\varepsilon u' \rangle + \langle a_2^\varepsilon u', f^{-2N} u'' \rangle + \langle a_2^\varepsilon u, f^{-2N} u''' \rangle - \langle f^{-2N} u'', a_2^\varepsilon u' \rangle \\
&\quad - \langle f^{-2N} u', a_2^\varepsilon u'' \rangle \\
&= \langle f^{-2N} u''', a_2^\varepsilon u \rangle + \langle a_2^\varepsilon u, f^{-2N} u''' \rangle + \langle [f^{-2N}, a_2^\varepsilon] u', u'' \rangle + \langle f^{-2N} \hat{\alpha}_1^\varepsilon u', u'' \rangle \\
&\quad = 2 \operatorname{Re} \langle f^{-2N} u''', a_2^\varepsilon u \rangle + \langle f^{-2N} \tilde{\alpha}_1^\varepsilon u', u'' \rangle.
\end{aligned}$$

To obtain the last line we used Corollary 5.1 and the fact that the principal symbol of  $a_2^\varepsilon$  is real, i.e.  $a_2^\varepsilon - (a_2^\varepsilon)^*$  is a first order operator.

**5.6. Estimate of  $I_7$ .** Let us rewrite  $iI_7$  in the following way

$$\begin{aligned} iI_7 &= 2i \operatorname{Im} \langle f^{-2N} \varepsilon^{1/3} t^2 a_3^\varepsilon u, u'' \rangle \\ &= \varepsilon^{1/3} t^2 \left( \langle f^{-2N} a_3^\varepsilon u, u'' \rangle - \langle u'', f^{-2N} a_3^\varepsilon u \rangle \right). \end{aligned}$$

We have the identity

$$\begin{aligned} \partial_t 2i \operatorname{Im} \langle f^{-2N} a_3^\varepsilon u, u' \rangle &= \langle f^{-2N} a_3^\varepsilon u, u'' \rangle - \langle u'', f^{-2N} a_3^\varepsilon u \rangle \\ &\quad + \langle f^{-2N} a_3^\varepsilon u', u' \rangle - \langle u', f^{-2N} a_3^\varepsilon u' \rangle \\ &\quad - 2N \left( \langle f^{-2N-1} a_3^\varepsilon u, u' \rangle - \langle u', f^{-2N-1} a_3^\varepsilon u \rangle \right) \\ &\quad + \langle f^{-2N} (\partial_t a_3^\varepsilon) u, u' \rangle - \langle u', f^{-2N} (\partial_t a_3^\varepsilon) u \rangle. \end{aligned} \tag{5.10}$$

Plugging (5.10) into the above expression for  $iI_7$ , we then obtain

$$\begin{aligned} I_7 &= \varepsilon^{1/3} t^2 \partial_t (2 \operatorname{Im} \langle f^{-2N} a_3^\varepsilon u, u' \rangle) \\ &\quad + \varepsilon^{1/3} t^2 2N 2 \operatorname{Im} \langle f^{-2N-1} a_3^\varepsilon u, u' \rangle \\ &\quad - \varepsilon^{1/3} t^2 2 \operatorname{Im} \langle f^{-2N} a_3^\varepsilon u', u' \rangle - \varepsilon^{1/3} t^2 2 \operatorname{Im} \langle f^{-2N} (\partial_t a_3^\varepsilon) u, u' \rangle \\ &= \varepsilon^{1/3} t^2 \partial_t (2 \operatorname{Im} \langle f^{-2N} a_3^\varepsilon u, u' \rangle) \\ &\quad + \varepsilon^{1/3} t^2 2N 2 \operatorname{Im} \langle f^{-2N-1} a_3^\varepsilon u, u' \rangle \\ &\quad + \sum_{k=1}^2 I_{7,k}. \end{aligned} \tag{5.11}$$

**5.7. Estimate of  $I_6$ .** In this section we manipulate the term  $I_6$ .

$$\begin{aligned} iI_6 &= -\varepsilon^{1/3} \theta t^2 \ 2i \operatorname{Im} \langle f^{-2N} a_1^\varepsilon \partial_t^2 u, a_2^\varepsilon u \rangle \\ &= \varepsilon^{1/3} \theta t^2 \ 2i \operatorname{Im} \langle f^{-2N} a_2^\varepsilon u, a_1^\varepsilon u'' \rangle = \varepsilon^{1/3} \theta t^2 \ 2i \operatorname{Im} \langle a_1^{\varepsilon*} f^{-2N} a_2^\varepsilon u, u'' \rangle \\ &= \varepsilon^{1/3} \theta t^2 \left\{ 2i \operatorname{Im} \langle f^{-2N} a_1^{\varepsilon*} a_2^\varepsilon u, u'' \rangle + 2i \operatorname{Im} \langle f^{-2N} \alpha_0^\varepsilon a_2^\varepsilon u, u'' \rangle \right\} \end{aligned} \tag{5.12}$$

Actually without any loss in generality we may assume that  $a_j$  are formally self adjoint differential operators; in fact their principal symbol is real, by the hyperbolicity assumption and hence it all amounts to a change in the lower order terms that can be easily handled, since it does not affect the hypotheses.

The first summand in the last line above is treated as we did for  $I_7$ , thus yielding

$$\begin{aligned} I_6 &= \varepsilon^{1/3} \theta t^2 \partial_t (2 \operatorname{Im} \langle f^{-2N} a_1^\varepsilon a_2^\varepsilon u, u' \rangle) \\ &\quad + \varepsilon^{1/3} \theta t^2 2N 2 \operatorname{Im} \langle f^{-2N-1} a_1^\varepsilon a_2^\varepsilon u, u' \rangle \\ &\quad - \varepsilon^{1/3} \theta t^2 \left\{ 2 \operatorname{Im} \langle f^{-2N} a_1^\varepsilon a_2^\varepsilon u', u' \rangle + 2 \operatorname{Im} \langle f^{-2N} \partial_t (a_1^\varepsilon a_2^\varepsilon) u, u' \rangle \right. \\ &\quad \left. + 2 \operatorname{Im} \langle f^{-2N} \alpha_0^\varepsilon a_2^\varepsilon u, u'' \rangle \right\}. \end{aligned} \tag{5.13}$$

So that eventually we may write

$$\begin{aligned}
I_6 &= \varepsilon^{1/3} \theta t^2 \partial_t \left( 2 \operatorname{Im} \langle f^{-2N} a_1^\varepsilon a_2^\varepsilon u, u' \rangle \right) \\
&\quad + \varepsilon^{1/3} \theta t^2 2N 2 \operatorname{Im} \langle f^{-2N-1} a_1^\varepsilon a_2^\varepsilon u, u' \rangle \\
&\quad + \sum_{k=1}^3 I_{6,k}.
\end{aligned} \tag{5.14}$$

**5.8. Estimate of  $I_8$ .** Using the calculus it is not difficult to show that there is a symbol of order zero,  $\hat{\alpha}_0^\varepsilon$ , such that

$$\begin{aligned}
I_8 &= 2\varepsilon^{1/3} \theta t^3 \operatorname{Im} \langle f^{-2N} a_3^\varepsilon u, a_2^\varepsilon u \rangle \\
&= \varepsilon^{1/3} \theta t^3 2 \operatorname{Im} \langle f^{-2N} a_3^\varepsilon u, a_2^\varepsilon u \rangle \\
&= \varepsilon^{1/3} \theta t^3 2 \operatorname{Im} \langle f^{-2N} \hat{\alpha}_0^\varepsilon a_2^\varepsilon u, a_2^\varepsilon u \rangle.
\end{aligned} \tag{5.15}$$

Here our argument is based on the fact that symbols of the operators  $a_2^\varepsilon(t, x, D_x)$  and  $a_3^\varepsilon(t, x, D_x)$  are real valued. Thus

$$(a_2^\varepsilon(t, x, D_x) a_3^\varepsilon(t, x, D_x))^* = a_2^\varepsilon(t, x, D_x) a_3^\varepsilon(t, x, D_x) + \alpha_4^\varepsilon(t, x, D_x)$$

with a pseudodifferential operator  $\alpha_4^\varepsilon$  of order 4. Since  $a_2^\varepsilon(t, x, D_x)$  is elliptic, it is easy to find  $\alpha_0^\varepsilon(t, x, d_x)$  so that  $\alpha_4^\varepsilon(t, x, D_x) = (a_2^\varepsilon)^* \alpha_0^\varepsilon(t, x, D_x) a_2^\varepsilon$ .

## 6. ENERGIES

So far we have obtained the following expression for the quantity on the r.h.s. of (4.8):

$$\begin{aligned}
-2 \operatorname{Im} \langle f^{-2N} \mathcal{P}u, Mu \rangle &= \partial_t \left[ \|f^{-N} u''\|^2 + (1-\theta)t \operatorname{Re} \langle f^{-2N} a_2^\varepsilon u', u' \rangle + \theta t^2 \|f^{-N} a_2^\varepsilon u\|^2 \right. \\
&\quad + \theta t 2 \operatorname{Re} \langle f^{-2N} a_2^\varepsilon u, u'' \rangle \\
&\quad \left. + \varepsilon^{1/3} t^2 2 \operatorname{Im} \langle f^{-2N} a_3^\varepsilon u, u' \rangle + \varepsilon^{1/3} \theta t^2 2 \operatorname{Im} \langle f^{-2N} a_1^\varepsilon a_2^\varepsilon u, u' \rangle \right] \\
&+ 2N \left[ \|f^{-N-1/2} u''\|^2 + (1-\theta)t \operatorname{Re} \langle f^{-2N-1} a_2^\varepsilon u', u' \rangle + \theta t^2 \|f^{-N-1/2} a_2^\varepsilon u\|^2 \right. \\
&\quad \left. + \theta t 2 \operatorname{Re} \langle f^{-2N-1} a_2^\varepsilon u, u'' \rangle \right] \\
&+ 2N \varepsilon^{1/3} t^2 2 \operatorname{Im} \langle f^{-2N-1} a_3^\varepsilon u, u' \rangle + 2N \varepsilon^{1/3} \theta t^2 2 \operatorname{Im} \langle f^{-2N-1} a_1^\varepsilon a_2^\varepsilon u, u' \rangle \\
&- (1-\theta) \operatorname{Re} \langle f^{-2N} a_2^\varepsilon u', u' \rangle - 2\theta \operatorname{Re} \langle f^{-2N} a_2^\varepsilon u, u'' \rangle - 2\theta t \|f^{-N} a_2^\varepsilon u\|^2 \\
&+ \theta t 2 \operatorname{Re} \langle f^{-2N} \tilde{\alpha}_1^\varepsilon u', u'' \rangle + 2 \operatorname{Re} \langle f^{-2N} u'', \alpha_1^\varepsilon u' \rangle \\
&- 2\theta \varepsilon^{2/3} t^2 \operatorname{Re} \langle f^{-2N} (\partial_t a_2)^\varepsilon u, a_2^\varepsilon u \rangle + \varepsilon^{1/3} t 2 \operatorname{Re} \langle f^{-2N} \alpha_0^\varepsilon u'', u'' \rangle \\
&- \theta \varepsilon^{2/3} t 2 \operatorname{Re} \langle f^{-2N} u'', (\partial_t a_2)^\varepsilon u \rangle - (1-\theta) t \varepsilon^{2/3} \langle f^{-2N} (\partial_t a_2)^\varepsilon u', u' \rangle \\
&- 2\varepsilon^{1/3} t 2 \operatorname{Im} \langle f^{-2N} a_3^\varepsilon u, u' \rangle - \varepsilon^{1/3} t^2 2 \operatorname{Im} \langle f^{-2N} a_3^\varepsilon u', u' \rangle - \varepsilon^{1/3} t^2 2 \operatorname{Im} \langle f^{-2N} (\partial_t a_3^\varepsilon) u, u' \rangle \\
&- 2\varepsilon^{1/3} \theta t 2 \operatorname{Im} \langle f^{-2N} a_1^\varepsilon a_2^\varepsilon u, u' \rangle - \varepsilon^{1/3} t^2 2 \operatorname{Im} \langle f^{-2N} a_1^\varepsilon a_2^\varepsilon u', u' \rangle \\
&- 2t^2 \varepsilon^{2/3} \operatorname{Im} \langle f^{-2N} (\partial_t (a_1^\varepsilon a_2^\varepsilon)) u, u' \rangle \\
&- \varepsilon^{1/3} t^2 \theta 2 \operatorname{Im} \langle f^{-2N} \alpha_0^\varepsilon a_2^\varepsilon u, u'' \rangle + \varepsilon^{1/3} t^3 2 \operatorname{Re} \langle f^{-2N} \hat{\alpha}_0^\varepsilon a_2^\varepsilon u, a_2^\varepsilon u \rangle \\
&+ 2 \operatorname{Im} \langle f^{-2N} b_2^\varepsilon(t, x, D_x) u, \partial_t^2 u \rangle + 2 \operatorname{Im} \langle f^{-2N} b_2^\varepsilon(t, x, D_x) u, \theta t a_2^\varepsilon u \rangle \\
&+ \text{lower order terms} \\
&= \partial_t \mathcal{E}_N + 2N \left[ \|f^{-N-1/2} u''\|^2 + (1-\theta)t \operatorname{Re} \langle f^{-2N-1} a_2^\varepsilon u', u' \rangle + \theta t^2 \|f^{-N-1/2} a_2^\varepsilon u\|^2 \right. \\
&\quad \left. + \theta t 2 \operatorname{Re} \langle f^{-2N-1} a_2^\varepsilon u, u'' \rangle \right] \\
&+ 2N \varepsilon^{1/3} t^2 2 \operatorname{Im} \langle f^{-2N-1} a_3^\varepsilon u, u' \rangle - 2N \varepsilon^{1/3} \theta t^2 2 \operatorname{Im} \langle f^{-2N-1} a_1^\varepsilon a_2^\varepsilon u, u' \rangle \\
&+ \sum_{j=1}^{19} F_j + \text{lower order terms}
\end{aligned} \tag{6.1}$$

We write  $a_2^\varepsilon(t, x, \xi) = a_2^\varepsilon(0, x, \xi) + \varepsilon^{2/3} t \tilde{a}_2^\varepsilon(t, x, \xi)$  and we get

$$(1-\theta)t \operatorname{Re} \langle f^{-2N-1} a_2^\varepsilon u', u' \rangle + \theta t^2 \|f^{-N-1/2} a_2^\varepsilon u\|^2 + \theta t 2 \operatorname{Re} \langle f^{-2N-1} a_2^\varepsilon u, u'' \rangle$$

$$\begin{aligned}
&= (1 - \theta)t \operatorname{Re} \langle f^{-2N-1} a_2^\varepsilon(0, x, D_x) u', u' \rangle + \theta t^2 \|f^{-N-1/2} a_2^\varepsilon(0, x, D_x) u\|^2 \\
&\quad + \theta t \operatorname{Re} \langle f^{-2N-1} a_2^\varepsilon(0, x, D_x) u, u'' \rangle + t^2 \varepsilon^{2/3} A_{N+1/2}^{(2)}(u).
\end{aligned}$$

Here  $t^2 \varepsilon^{2/3} A_{N+1/2}^{(3)}(u)$  denotes terms which have coefficient  $t^2 \varepsilon^{2/3}$  and it will be easy to absorb them by applying the following

**Proposition 6.1.** *There exist positive constants  $C_1$  and  $C_2$ , independent of the positive integer  $k$  such that for  $0 < \varepsilon \leq \varepsilon_0(k)$  we have*

$$\langle f^{-k} a_2^\varepsilon(0, x, D_x) v, f^{-k} v \rangle \geq C_1 \|f^{-k} v\|_1^2 - C_2 \|f^{-k} v\|_0^2, \quad (6.2)$$

for every  $v \in C_0^\infty$ . The constant  $C_1$  depends only on the symbol  $a_2^\varepsilon(0, x, \xi)$ .

*Proof.* The proof consists in just making sure that we may commute the weight operator  $f^{-k}$  with  $a_2^\varepsilon$  and estimate the errors, which naturally depend on  $N$ . We have that

$$\langle f^{-k} a_2^\varepsilon(0, x, D_x) v, f^{-k} v \rangle = \langle a_2^\varepsilon(0, x, D_x) f^{-k} v, f^{-k} v \rangle + \langle [f^{-k}, a_2^\varepsilon(0, x, D_x)] v, f^{-k} v \rangle = I_1 + I_2.$$

Keeping in mind that  $a_2^\varepsilon$  is uniformly elliptic and using the strict Gårding inequality for it, we obtain that

$$I_1 \geq c_1 \|f^{-k} v\|_1^2 - c_2 \|f^{-k} v\|_0^2,$$

for two suitable positive constants  $c_1$  and  $c_2$  independent of  $k$ . We are thus left with  $I_2$ . By Proposition 5.2 and Corollary 5.1 we see that if  $\varepsilon$  is small enough depending on  $k$ , i.e. if  $\varepsilon \leq \varepsilon_0(k)$ , there is a positive constant  $c_3$  independent on  $k$ , such that

$$|I_2| \leq c_3 \|f^{-k} v\|_{1/2}^2 \leq \delta \|f^{-k} v\|_1^2 + c'_3 \delta^{-1} \|f^{-k} v\|_0^2.$$

Choosing  $\delta$  conveniently small, but independent of  $\varepsilon$  and  $k$  we obtain the assertion.  $\square$

To simplify the notations in the following we will write  $\mathbf{a}_2^\varepsilon$  for  $a_2^\varepsilon(0, x, D_x)$ , while  $a_2^\varepsilon$  will denote the operator  $a_2^\varepsilon(t, x, D_x)$ . Now we introduce the energy by the following

**Definition 6.1.** *For a non negative integer  $k$  we define the  $k$ -th energy as*

$$E_k(u) = \frac{1}{3} \left( \|f^{-k} u''\|^2 + 2t \operatorname{Re} \langle f^{-k} \mathbf{a}_2^\varepsilon u', f^{-k} u' \rangle + \frac{1}{2} t^2 \|f^{-k} \mathbf{a}_2^\varepsilon u\|^2 \right).$$

Consider the term in the brackets  $\left[ \dots \right]$  with  $2N$  in front in equation (6.1). We write  $a_2^\varepsilon(t, x, \xi) = a_2^\varepsilon(0, x, \xi) + \varepsilon^{2/3} t \tilde{a}_2^\varepsilon(t, x, \xi)$  and we keep only the terms with  $a_2^\varepsilon(0, x, D_x)$ . The sum of terms with a factor  $t^2 \varepsilon^{2/3}$  after this splitting has been denoted by  $t^2 \varepsilon^{2/3} A_{N+1/2}^{(2)}(u)$ . For a small  $\eta > 0$  we have

$$\theta t \operatorname{Re} \langle f^{-2k} \mathbf{a}_2^\varepsilon u, u'' \rangle \leq \eta \|f^{-k} u''\|^2 + \frac{\theta^2}{\eta} t^2 \|f^{-k} \mathbf{a}_2^\varepsilon u\|^2,$$

We take  $\theta = \frac{1}{3}$ ,  $\eta = \frac{2}{3}$  and get  $\frac{1}{3} - \frac{3}{18} = \frac{1}{6}$ . Then

$$\begin{aligned}
&\|f^{-N-1/2} u''\|^2 + (1 - \theta)t \operatorname{Re} \langle f^{-2N-1} \mathbf{a}_2^\varepsilon u', u' \rangle + \theta t^2 \|f^{-N-1/2} \mathbf{a}_2^\varepsilon u\|^2 + \theta t \operatorname{Re} \langle f^{-2N-1} \mathbf{a}_2^\varepsilon u, u'' \rangle \\
&\geq \frac{1}{3} \|f^{-N-1/2} u''\|^2 + \frac{2}{3} t \operatorname{Re} \langle f^{-2N-1} \mathbf{a}_2^\varepsilon u', u' \rangle + \frac{t^2}{6} \|f^{-N-1/2} \mathbf{a}_2^\varepsilon u\|^2 = E_{N+1/2}(u).
\end{aligned}$$

Going back to the operator  $\mathcal{P}$ , we have

$$\begin{aligned} 2 \operatorname{Im} \langle f^{-2N} \mathcal{P}u, Mu \rangle &= 2 \operatorname{Im} \langle f^{-2N} \mathcal{P}u, D_t^2 u \rangle + 2 \operatorname{Im} \langle f^{-2N} \mathcal{P}u, \theta t a_2^\varepsilon u \rangle \\ &\leq 2 \|f^{-N+\frac{1}{2}} \mathcal{P}u\|^2 + \|f^{-N-\frac{1}{2}} \partial_t^2 u\|^2 + t^2 \theta^2 \|f^{-N-1/2} a_2^\varepsilon u\|^2. \end{aligned}$$

We denote by  $\lambda > 0$  a large parameter and multiply both sides by  $e^{-\lambda t}$ . Since  $e^{-\lambda t} \partial_t \mathcal{E}_N(u) = \partial_t (e^{-\lambda t} \mathcal{E}_N(u)) + \lambda e^{-\lambda t} \mathcal{E}_N(u)$  we obtain from (6.1)

$$\begin{aligned} 2e^{-\lambda t} \|f^{-N+\frac{1}{2}} \mathcal{P}u\|^2 &\geq \partial_t \left( e^{-\lambda t} \mathcal{E}_N(u) \right) + \lambda e^{-\lambda t} \mathcal{E}_N(u) + 2Ne^{-\lambda t} E_{N+1/2}(u) \\ &\quad + 2Ne^{-\lambda t} t^2 \varepsilon^{2/3} A_{N+1/2}^{(2)}(u) \\ &\quad - e^{-\lambda t} \|f^{-N-\frac{1}{2}} \partial_t^2 u\|^2 - t^2 e^{-\lambda t} \theta^2 \|f^{-N-1/2} a_2^\varepsilon u\|^2 \\ &\quad + e^{-\lambda t} \sum_{j=1}^{19} F_j + \text{lower order terms.} \end{aligned} \tag{6.3}$$

Note that the splitting of the l.h.s. in (6.1) has generated two terms involving the *weight*  $f^{-N-1/2}$  and that these terms are actually present in the principal positive part of  $E_{N+1/2}(u)$ . Therefore they can be easily absorbed by  $E_{N+1/2}(u)$  due to the large constant in front of it. This fact however shows that the energy estimate (6.3) is still incomplete, since we are going to need terms with different powers of  $f$ . On the other hand, we can perform inside the term  $\mathcal{E}_N(u)$  in the same substitution we did for  $\mathcal{E}_{N+1/2}(u)$ —i.e. write  $a_2^\varepsilon(t, x, \xi) = a_2^\varepsilon(0, x, \xi) + \varepsilon^{2/3} t \tilde{a}_2^\varepsilon(t, x, \xi)$ —and we have trivially that

$$\mathcal{E}_N(u) \geq E_N(u) + \left( t^2 \varepsilon^{2/3} A_N^{(2)}(u) + t^2 \varepsilon^{1/3} A_N^{(3)}(u) \right),$$

where by  $t^2 \varepsilon^{1/3} A_N^{(3)}(u)$  we denoted the fifth and sixth summand in the expression of  $\mathcal{E}_N$ .

Let us now consider the following identity, where  $k$  is a positive integer and  $g$  denotes a smooth function in the same class as  $u$ :

$$e^{-\lambda t} f^{-2k} 2 \operatorname{Re} g' \bar{g} = \partial_t \left( e^{-\lambda t} f^{-2k} |g|^2 \right) + \lambda e^{-\lambda t} f^{-2k} |g|^2 + 2k e^{-\lambda t} f^{-2k-1} |g|^2.$$

This implies

$$e^{-\lambda t} f^{-2k+1} |g'|^2 \geq \partial_t \left( e^{-\lambda t} f^{-2k} |g|^2 \right) + \lambda e^{-\lambda t} f^{-2k} |g|^2 + (2k-1) e^{-\lambda t} f^{-2k-1} |g|^2.$$

Now, taking  $g = \partial_t u$ , we have

$$e^{-\lambda t} f^{-2k+1} |\partial_t^2 u|^2 \geq \partial_t \left( e^{-\lambda t} f^{-2k} |\partial_t u|^2 \right) + \lambda e^{-\lambda t} f^{-2k} |\partial_t u|^2 + (2k-1) e^{-\lambda t} f^{-2k-1} |\partial_t u|^2, \tag{6.4}$$

while, taking  $g = u$ , we get

$$e^{-\lambda t} f^{-2k+1} |\partial_t u|^2 \geq \partial_t \left( e^{-\lambda t} f^{-2k} |u|^2 \right) + \lambda e^{-\lambda t} f^{-2k} |u|^2 + (2k-1) e^{-\lambda t} f^{-2k-1} |u|^2. \tag{6.5}$$

From (6.4) and (6.5) above we obtain the following inequality

$$\begin{aligned}
e^{-\lambda t} \|f^{-k+\frac{1}{2}} \partial_t^2 u\|^2 &\geq \partial_t \left( e^{-\lambda t} \|f^{-k} \partial_t u\|^2 \right) + \lambda e^{-\lambda t} \|f^{-k} \partial_t u\|^2 \\
&\quad + (2k-2) e^{-\lambda t} \|f^{-k-\frac{1}{2}} \partial_t u\|^2 \\
&\quad + \partial_t \left( e^{-\lambda t} \|f^{-k-1} u\|^2 \right) + \lambda e^{-\lambda t} \|f^{-k-1} u\|^2 \\
&\quad + (2k+1) e^{-\lambda t} \|f^{-k-\frac{3}{2}} u\|^2.
\end{aligned} \tag{6.6}$$

Choose now  $k = N + 1$ . We obtain

$$\begin{aligned}
e^{-\lambda t} \|f^{-N-\frac{1}{2}} \partial_t^2 u\|^2 &\geq \partial_t \left( e^{-\lambda t} \|f^{-N-1} \partial_t u\|^2 \right) + \lambda e^{-\lambda t} \|f^{-N-1} \partial_t u\|^2 \\
&\quad + 2N e^{-\lambda t} \|f^{-N-\frac{3}{2}} \partial_t u\|^2 \\
&\quad + \partial_t \left( e^{-\lambda t} \|f^{-N-2} u\|^2 \right) + \lambda e^{-\lambda t} \|f^{-N-2} u\|^2 \\
&\quad + (2N+3) e^{-\lambda t} \|f^{-N-\frac{5}{2}} u\|^2.
\end{aligned} \tag{6.7}$$

We can rewrite (6.3) as follows

$$\begin{aligned}
2e^{-\lambda t} \|f^{-N+\frac{1}{2}} \mathcal{P}u\|^2 &\geq \partial_t \left( e^{-\lambda t} \mathcal{E}_N(u) \right) + \lambda e^{-\lambda t} E_N(u) + 2N e^{-\lambda t} E_{N+1/2}(u) \\
&\quad + t^2 e^{-\lambda t} \left[ 2N \left( \varepsilon^{2/3} A_{N+1/2}^{(2)}(u) + \varepsilon^{1/3} A_{N+1/2}^{(3)}(u) \right) \right. \\
&\quad \left. + \lambda \left( \varepsilon^{2/3} A_N^{(2)}(u) + \varepsilon^{1/3} A_N^{(3)}(u) \right) \right] \\
&\quad - e^{-\lambda t} \|f^{-N-\frac{1}{2}} \partial_t^2 u\|^2 - t^2 e^{-\lambda t} \theta^2 \|f^{-N-1/2} a_2^\varepsilon u\|^2 \\
&\quad + e^{-\lambda t} \sum_{j=1}^{19} F_j + \text{lower order terms} \\
&\geq \partial_t \left( e^{-\lambda t} \mathcal{E}_N(u) \right) + \lambda e^{-\lambda t} E_N(u) \\
&\quad + N e^{-\lambda t} \left[ \frac{1}{2} \|f^{-N-1/2} u''\|_0^2 + \frac{2}{3} t \operatorname{Re} \langle f^{-N-1/2} \mathbf{a}_2^\varepsilon u', f^{-N-1/2} u' \rangle \right. \\
&\quad \left. + \frac{1}{3} \|f^{-N-1/2} \mathbf{a}_2^\varepsilon u\|^2 \right] \\
&\quad + N e^{-\lambda t} \left( \frac{1}{6} \|f^{-N-1/2} u''\|^2 + \frac{2}{3} t \operatorname{Re} \langle f^{-N-1/2} \mathbf{a}_2^\varepsilon u', f^{-N-1/2} u' \rangle \right) \\
&\quad + e^{-\lambda t} \sum_{j=1}^{19} F_j + e^{-\lambda t} \sum_{j=1}^6 G_j + \text{lower order terms},
\end{aligned} \tag{6.8}$$

where we introduced the notation

$$\begin{aligned} \sum_{j=1}^6 G_j &= -\|f^{-N-\frac{1}{2}}\partial_t^2 u\|^2 - t^2\theta^2\|f^{-N-1/2}a_2^\varepsilon u\|^2 \\ &\quad + t^2\left[2N\left(\varepsilon^{2/3}A_{N+1/2}^{(2)}(u) + \varepsilon^{1/3}A_{N+1/2}^{(3)}(u)\right) \right. \\ &\quad \left. + \lambda\left(\varepsilon^{2/3}A_N^{(2)}(u) + \varepsilon^{1/3}A_N^{(3)}(u)\right)\right] \end{aligned} \quad (6.9)$$

Applying Proposition 6.1, we obtain

$$\operatorname{Re}\langle f^{-N-1/2}\mathbf{a}_2^\varepsilon u', f^{-N-1/2}u' \rangle \geq C_1\|f^{-N-1/2}u'\|_1^2 - C_2\|f^{-N-1/2}u'\|_0^2. \quad (6.10)$$

From (6.5) we deduce that for  $k \in \mathbb{N} \cup \{0\}$ ,

$$e^{-\lambda t}\|f^{-k+\frac{1}{2}}\partial_t u\|_1^2 \geq \partial_t \left( e^{-\lambda t}\|f^{-k}u\|_1^2 \right) + \lambda e^{-\lambda t}\|f^{-k}u\|_1^2 + (2k-1)e^{-\lambda t}\|f^{-k-\frac{1}{2}}u\|_1^2.$$

Choosing  $k = N+1$  we obtain

$$\begin{aligned} e^{-\lambda t}\|f^{-N-\frac{1}{2}}\partial_t u\|_1^2 &\geq \partial_t \left( e^{-\lambda t}\|f^{-N-1}u\|_1^2 \right) + \lambda e^{-\lambda t}\|f^{-N-1}u\|_1^2 \\ &\quad + (2N+1)e^{-\lambda t}\|f^{-N-\frac{3}{2}}u\|_1^2. \end{aligned} \quad (6.11)$$

This implies

$$\begin{aligned} te^{-\lambda t}\|f^{-N-\frac{1}{2}}\partial_t u\|_1^2 &\geq \partial_t \left( te^{-\lambda t}\|f^{-N-1}u\|_1^2 \right) - e^{-\lambda t}\|f^{-N-1}u\|_1^2 + t\lambda e^{-\lambda t}\|f^{-N-1}u\|_1^2 \\ &\quad + (2N+1)te^{-\lambda t}\|f^{-N-\frac{3}{2}}u\|_1^2. \end{aligned} \quad (6.12)$$

To treat the negative term in the right hand side, we apply the following lemma which will play a key role in the next section.

**Lemma 6.1.** *For  $t \geq 0$  and  $\xi \in \mathbb{R}^n$  we have*

$$\frac{1}{\langle \xi \rangle^2} + tf^2 \geq f^3. \quad (6.13)$$

*Proof.* The proof is a simple verification. In fact  $f^3 = f^2t/3 + f^2\langle \xi \rangle^{-2/3}$ . The latter quantity is equal to  $f^2t/3 + \langle \xi \rangle^{-2} + \frac{2}{3}t\langle \xi \rangle^{-4/3} + \frac{t^2}{9}\langle \xi \rangle^{-2/3}$ . It is clear that

$$t\left[\frac{2}{3}\langle \xi \rangle^{-4/3} + \frac{t}{9}\langle \xi \rangle^{-2/3}\right] \leq \frac{2}{3}tf^2$$

and this accomplishes the proof.  $\square$

The above Lemma implies

$$\langle \xi \rangle^2 f^{-2N-2} \leq t\langle \xi \rangle^2 f^{-2N-3} + f^{-2N-5},$$

hence

$$-\|f^{-N-1}u\|_1^2 + t\|f^{-N-3/2}u\|_1^2 \geq -\|f^{-N-5/2}u\|_0^2$$

so that (6.12) can be rewritten as

$$\begin{aligned} te^{-\lambda t}\|f^{-N-\frac{1}{2}}\partial_t u\|_1^2 &\geq \partial_t \left( te^{-\lambda t}\|f^{-N-1}u\|_1^2 \right) + t\lambda e^{-\lambda t}\|f^{-N-1}u\|_1^2 \\ &\quad + 2Nte^{-\lambda t}\|f^{-N-\frac{3}{2}}u\|_1^2 - \|f^{-N-5/2}u\|_0^2. \end{aligned} \quad (6.14)$$



Next, taking into account the inequalities (6.7), (6.10) and (6.14), we get

$$\begin{aligned}
& \frac{1}{6}e^{-\lambda t}\|f^{-N-1/2}u''\|^2 + \frac{2}{3}te^{-\lambda t}\operatorname{Re}\langle f^{-N-1/2}\mathbf{a}_2^\varepsilon u', f^{-N-1/2}u' \rangle \\
& \geq \frac{1}{6}\left[\partial_t\left(e^{-\lambda t}\|f^{-N-1}u'\|_0^2\right) + \lambda e^{-\lambda t}\|f^{-N-1}u'\|_0^2\right. \\
& \quad + 2Ne^{-\lambda t}\|f^{-N-\frac{3}{2}}u'\|^2 + \partial_t\left(e^{-\lambda t}\|f^{-N-2}u\|_0^2\right) + \lambda e^{-\lambda t}\|f^{-N-2}u\|_0^2 \\
& \quad \left. + (2N+3-2C_1)e^{-\lambda t}\|f^{-N-\frac{5}{2}}u\|_0^2\right] \\
& + \frac{1}{3}C_1\left[\partial_t\left(te^{-\lambda t}\|f^{-N-1}u\|_1^2\right) + \lambda te^{-\lambda t}\|f^{-N-1}u\|_1^2 + 2Nte^{-\lambda t}\|f^{-N-\frac{3}{2}}u\|_1^2\right] \\
& + \frac{1}{3}C_1t\|f^{-N-1/2}u'\|_1^2 - \frac{2}{3}C_2te^{-\lambda t}\|f^{-N-1/2}u'\|_0^2. \quad (6.15)
\end{aligned}$$

The last term on the right hand side can be absorbed for small  $t$  by the term  $\lambda e^{-\lambda t}\|f^{-N-1}u'\|_0^2$  since  $f^{1/2} \leq c$ . Also by using the calculus of pseudodifferential operators, Corollary 5.1 and Proposition 6.1, we may write

$$\begin{aligned}
2t\operatorname{Re}\langle f^{-N-1/2}\mathbf{a}_2^\varepsilon u', f^{-N-1/2}u' \rangle & \geq 2t\operatorname{Re}\langle \mathbf{a}_2^\varepsilon f^{-N-1/2}u', f^{-N-1/2}u' \rangle - C_3t\|f^{-N-1/2}u'\|_{1/2}^2 \\
& \geq t\operatorname{Re}\langle \mathbf{a}_2^\varepsilon f^{-N-1/2}u', f^{-N-1/2}u' \rangle - C_4t\|f^{-N-1/2}u'\|_0^2.
\end{aligned}$$

For small  $t$  we may absorb the term with  $C_4t$  in front as we did above, taking  $\lambda$  large. Eventually, modulo lower order terms which can be absorbed by terms with a large  $\lambda$  in (6.15) we have

$$t^2\|f^{-N-1/2}\mathbf{a}_2^\varepsilon u\|_0^2 = t^2\|\mathbf{a}_2^\varepsilon f^{-N-1/2}u\|_0^2.$$

Therefore, using (6.15), we have for small  $t$  and large  $\lambda$  the estimate

$$\begin{aligned}
& Ne^{-\lambda t}\left[\frac{1}{2}\|f^{-N-1/2}u''\|_0^2 + \frac{2}{3}t\operatorname{Re}\langle f^{-N-1/2}\mathbf{a}_2^\varepsilon u', f^{-N-1/2}u' \rangle + \frac{1}{3}t^2\|f^{-N-1/2}\mathbf{a}_2^\varepsilon u\|_0^2\right] \\
& + Ne^{-\lambda t}\left(\frac{1}{6}\|f^{-N-1/2}u''\|_0^2 + \frac{2}{3}t\operatorname{Re}\langle f^{-N-1/2}\mathbf{a}_2^\varepsilon u', f^{-N-1/2}u' \rangle\right) \\
& \geq \partial_t\left(\frac{N}{6}e^{-\lambda t}\|f^{-N-1}u'\|_0^2 + \frac{N}{6}e^{-\lambda t}\|f^{-N-2}u\|_0^2 + \frac{N}{3}C_1te^{-\lambda t}\|f^{-N-1}u\|_1^2\right) \\
& + Ne^{-\lambda t}\left[\frac{1}{2}\|f^{-N-1/2}u''\|_0^2 + \frac{1}{3}t\operatorname{Re}\langle \mathbf{a}_2^\varepsilon f^{-N-1/2}u', f^{-N-1/2}u' \rangle + \frac{1}{3}t^2\|\mathbf{a}_2^\varepsilon f^{-N-1/2}u\|_0^2\right] \\
& + \frac{N^2}{3}e^{-\lambda t}\left[\|f^{-N-3/2}u'\|_0^2 + \frac{2N+3-2C_1}{2N}\|f^{-N-5/2}u\|_0^2 + 2C_1te^{-\lambda t}\|f^{-N-3/2}u\|_1^2\right] \\
& + \frac{1}{3}NC_1te^{-\lambda t}\|f^{-N-1/2}u'\|_1^2 \\
& + \lambda Ne^{-\lambda t}\left\{\frac{1}{6}\|f^{-N-2}u\|_0^2 + \frac{1}{12}\|f^{-N-1}u'\|_0^2 + \frac{1}{6}C_1t\|f^{-N-1}u\|_1^2\right\}. \quad (6.16)
\end{aligned}$$

Here in the last two terms on the right hand side we replaced  $\lambda$  by  $\lambda/2$  to take into account the terms than have been absorbed with large  $\lambda$ .

Finally, we obtain the energy estimate

$$\begin{aligned}
e^{-\lambda t} \|f^{-N+\frac{1}{2}} \mathcal{P}u\|^2 &\geq \partial_t \left( e^{-\lambda t} \mathcal{E}_N(u) + \frac{N}{6} e^{-\lambda t} \|f^{-N-1} u'\|_0^2 \right. \\
&\quad \left. + \frac{N}{6} e^{-\lambda t} \|f^{-N-2} u\|_0^2 + \frac{N}{3} C_1 t e^{-\lambda t} \|f^{-N-1} u\|_1^2 \right) + \lambda e^{-\lambda t} E_N(u) \\
&\quad + N e^{-\lambda t} \left[ \frac{1}{2} \|f^{-N-1/2} u''\|_0^2 + \frac{1}{3} t \operatorname{Re} \langle \mathbf{a}_2^\varepsilon f^{-N-1/2} u', f^{-N-1/2} u' \rangle + \frac{1}{3} t^2 \|\mathbf{a}_2^\varepsilon f^{-N-1/2} u\|_0^2 \right] \\
&\quad + \frac{N^2}{3} e^{-\lambda t} \left[ \|f^{-N-3/2} u'\|_0^2 + \frac{2N+3-2C_1}{2N} \|f^{-N-5/2} u\|_0^2 + 2C_1 t e^{-\lambda t} \|f^{-N-3/2} u\|_1^2 \right] \\
&\quad + \frac{1}{3} N C_1 t e^{-\lambda t} \|f^{-N-1/2} u'\|_1^2 \\
&\quad + \lambda N e^{-\lambda t} \left\{ \frac{1}{12} \|f^{-N-1} u'\|_0^2 + \frac{1}{6} \|f^{-N-2} u\|_0^2 + \frac{1}{6} C_1 t \|f^{-N-1} u\|_1^2 \right\} \\
&\quad + e^{-\lambda t} \sum_{j=1}^{19} F_j + e^{-\lambda t} \sum_{j=1}^6 G_j + \text{lower order terms.} \tag{6.17}
\end{aligned}$$

Note that the  $F_j$  have been defined in (6.1), while the  $G_j$  are defined in (6.9).

## 7. ESTIMATE OF THE ERROR TERMS IN THE ENERGY INEQUALITY

The last line of (6.17) contains two sums grouping the “error” terms that must be dominated with the other positive terms. We name them according to (6.9) and (6.1). We point out that the last couple of the first group of terms are just those associated to the lower order terms containing pure second order  $x$ -derivatives and did not play any role up to now. The terms  $F_1, F_2, F_3, F_{18}, F_{19}$  contain the main contributions. It is convenient for the terms  $F_1, F_2, F_3$  to write  $a_2^\varepsilon(t, x, \xi) = a_2^\varepsilon(0, x, \xi) + t\varepsilon^{2/3} \tilde{a}_2^\varepsilon(t, x, \xi)$  and to replace the operator  $a_2^\varepsilon(t, x, D_x)$  by the operator  $\mathbf{a}_2^\varepsilon$  with symbol  $a_2^\varepsilon(0, x, \xi)$ . This will add a few terms similar to  $t\varepsilon^{2/3} A_N^{(2)}(u)$  but now we do not have a large coefficient  $\lambda$  as in  $G_5$  and  $G_6$ . Consequently, we have to deal with lower order terms which can only be treated choosing  $\varepsilon$  small, much in the same way as we treat  $G_3 \dots G_6$  below in this section. Next, using the calculus of pseudodifferential operators we write

$$\begin{aligned}
&-\frac{2}{3} [\operatorname{Re} \langle f^{-2N} \mathbf{a}_2^\varepsilon u', u' \rangle + \operatorname{Re} \langle f^{-2N} \mathbf{a}_2^\varepsilon u, u'' \rangle + t \|f^{-N} \mathbf{a}_2^\varepsilon u\|^2] \\
&= -\frac{2}{3} [\operatorname{Re} \langle \mathbf{a}_2^\varepsilon f^{-N} u', f^{-N} u' \rangle + \operatorname{Re} \langle \mathbf{a}_2^\varepsilon f^{-N} u, f^{-N} u'' \rangle + t \|\mathbf{a}_2^\varepsilon f^{-N} u\|^2] + A_1 + A_2 + A_3 = \sum_{j=1}^3 \tilde{F}_j + \sum_{j=1}^3 A_j.
\end{aligned}$$

The terms  $A_j, j = 1, 2, 3$  contain commutators and hence lower order operators. Thus they are easily treated. For example,

$$|A_1| = \frac{2}{3} |\operatorname{Re} \langle ([f^{-N} \mathbf{a}_2^\varepsilon] f^N) f^{-N} u', f^{-N} u' \rangle| \leq M_1 \|f^{-N} u'\|_{1/2}^2$$

since  $[f^{-N} \mathbf{a}_2^\varepsilon] f^N$  is a first order operator according to Corollary 5.1. Next, by Lemma 6.1 we have

$$\langle \xi \rangle f^{-2N} \leq \delta \langle \xi \rangle^2 f^{-2N} + D_\delta f^{-2N} \leq t \delta \langle \xi \rangle^2 f^{-2N-1} + \delta f^{-2N-3} + D_\delta f^{-2N} \tag{7.1}$$

where  $D_\delta$  denotes a suitable constant, large when  $\delta$  is small. We obtain

$$M_1 \|f^{-N} u'\|_{1/2}^2 \leq M_1 \delta t \|f^{-N-1/2} u'\|_1^2 + M_1 \delta \|f^{-N-3/2} u'\|_0^2 + M_1 D_\delta \|f^{-N} u'\|_0^2.$$

For small  $\delta$  the first term on the right hand side can be absorbed by  $\frac{1}{3}NC_1t\|f^{-N-1/2}u'\|_1^2$ , while the second one can be absorbed by  $\frac{N^2}{3}\|f^{-N-3/2}u'\|_0^2$  in (6.17). The last term can be absorbed choosing  $\lambda$  large by  $\frac{1}{12}\lambda N\|f^{-N-1}u'\|_0^2$ . For  $A_2$  we apply the inequality

$$|A_2| \leq \delta\|f^{-N}u\|_1^2 + \frac{1}{\delta}\|f^{-N}u''\|_0^2.$$

The term  $\frac{1}{\delta}\|f^{-N}u''\|_0^2$  can be absorbed by  $\lambda E_N(u)$  taking  $\lambda$  large. For  $\delta\|f^{-N}u\|_1^2$  we apply an argument similar to that used above. Finally, to deal with  $A_3$  we observe that  $|A_3| \leq M_3t\|f^{-N}u\|_1^2$  and this term can be easily absorbed by  $\frac{1}{6}\lambda NC_1t\|f^{-N-1}u\|_1^2$ . Notice that for the terms  $A_1, A_2, A_3$  it is not necessary to choose  $N$  large.

For the analysis of the terms  $F_{18}$  and  $F_{19}$  we apply a similar procedure. First we write  $b_2^\varepsilon(t, x, \xi) = b_2^\varepsilon(0, x, \xi) + t\varepsilon^{2/3}\tilde{b}_2^\varepsilon(t, x, \xi)$ . The terms with the factor  $t\varepsilon^{2/3}$  are similar to  $t\varepsilon^{2/3}A_N^{(2)}(u)$  and can be treated choosing  $\varepsilon$  small. We are going to do this below for  $t\varepsilon^{2/3}A_N^{(2)}(u)$ . To keep the notation simple we denote by  $\mathbf{b}_2^\varepsilon$  the operator with symbol  $b_2^\varepsilon(0, x, \xi)$ . The modified terms  $F_{18}, F_{19}$  will be denoted by  $\tilde{F}_{18}, \tilde{F}_{19}$ .

**7.1. Analysis of the term  $\tilde{F}_1$ .** We have

$$e^{-\lambda t}\tilde{F}_1 = -\frac{2}{3}e^{-\lambda t}\operatorname{Re}\langle \mathbf{a}_2^\varepsilon f^{-N}\partial_t u, f^{-N}\partial_t u \rangle.$$

According to Lemma 6.1, we have the inequality

$$\mathbf{a}_2^\varepsilon(0, x, \xi) \leq t\mathbf{a}_2^\varepsilon(0, x, \xi)f^{-1} + \alpha_0(0, x, \xi)f^{-3}, \quad (7.2)$$

where  $\alpha_0 = \frac{a_2^\varepsilon(0, x, \xi)}{(\xi)^2}$ . Now we take from (6.17) the term

$$\begin{aligned} H_1 &= \frac{2}{3}t\operatorname{Re}\langle \mathbf{a}_2^\varepsilon f^{-N-1/2}\partial_t u, f^{-N-1/2}\partial_t u \rangle = \frac{2}{3}t\operatorname{Re}\langle \mathbf{a}_2^\varepsilon f^{-N-1}\partial_t u, f^{-N}\partial_t u \rangle \\ &\quad + \frac{2}{3}t\operatorname{Re}\langle f^{1/2}[f^{-1/2}, \mathbf{a}_2^\varepsilon]f^{-N-1/2}\partial_t u, f^{-N-1/2}\partial_t u \rangle = \tilde{H}_1 + R_1. \end{aligned}$$

By the calculus of pseudodifferential operators  $f^{1/2}[f^{-1/2}, \mathbf{a}_2^\varepsilon]$  is a first order operator and we have

$$|R_1| \leq C_6t\|f^{-N-1/2}\partial_t u\|_{1/2}^2 \leq C_6\delta t\|f^{-N-1/2}\partial_t u\|_1^2 + C_6tD_\delta\|f^{-N-1/2}\partial_t u\|_0^2.$$

For small  $\delta$  we may absorb the term with coefficient  $C_6\delta t$  without choosing  $N$  large, while the second term with  $C_6tD_\delta$  can be absorbed for small  $t$  or large  $\lambda$ . Thus we must study

$$e^{-\lambda t}(\tilde{F}_1 + \tilde{H}_1) = \frac{2}{3}e^{-\lambda t}\operatorname{Re}\langle (t\mathbf{a}_2^\varepsilon f^{-1} - \mathbf{a}_2^\varepsilon)f^{-N}\partial_t u, f^{-N}\partial_t u \rangle.$$

We observe that  $t\mathbf{a}_2^\varepsilon(0, x, \xi)f^{-1}$  is a symbol in  $S(\langle \xi \rangle^2, g^\varepsilon)$ , since  $tf^{-1}$  is a symbol of order 0. Also,  $\alpha_0 f^{-3} \in S(\langle \xi \rangle^2, g^\varepsilon)$ . By using (7.2) and the sharp Gårding inequality, we get

$$\frac{2}{3}\operatorname{Re}\langle (t\mathbf{a}_2^\varepsilon f^{-1} - \mathbf{a}_2^\varepsilon + \alpha_0 f^{-3})f^{-N}\partial_t u, f^{-N}\partial_t u \rangle \geq -C_0\|f^{-N}\partial_t u\|_{1/2}^2,$$

hence

$$e^{-\lambda t}(\tilde{F}_1 + \tilde{H}_1) \geq -e^{-\lambda t}C_0\|f^{-N}\partial_t u\|_{1/2}^2 - A_1e^{-\lambda t}\|f^{-N-3/2}\partial_t u\|_0^2,$$

where  $C_0 > 0$  and  $A_1 > 0$  depend on  $a_2^\varepsilon(0, x, \xi)$  and we have used the fact that  $f^{3/2}\alpha_0 f^{-3/2}$  is a zero order operator. The first term on the right hand side can be treated as above exploiting (7.1). the second one can be absorbed by  $\frac{1}{3}N^2e^{-\lambda t}\|f^{-N-3/2}u'\|_0^2$  in (6.17).

**7.2. Analysis of the term  $\tilde{F}_3$ .** We have

$$\tilde{F}_3 = -\frac{2}{3}t \operatorname{Re}\langle (\mathbf{a}_2^\varepsilon)^2 f^{-N}u, f^{-N}u \rangle + R_3,$$

where  $|R_3| \leq C_7 t \|f^{-N}u\|_{3/2}^2$  and  $(\mathbf{a}_2^\varepsilon)^2$  denotes a pseudodifferential operator with symbol  $(a_2^\varepsilon)^2(0, x, \xi)$ . We multiply (7.2) by  $a_2^\varepsilon(0, x, \xi)$  and we get

$$(a_2^\varepsilon)^2 \leq t(a_2^\varepsilon)^2 f^{-1} + a_2^\varepsilon \alpha_0 f^{-3}.$$

Consider again the term  $\tilde{H}_1$ . Then

$$e^{-\lambda t}(\tilde{F}_3 + \tilde{H}_1) = \frac{2}{3}e^{-\lambda t}t \operatorname{Re}\langle (t(a_2^\varepsilon)^2 f^{-1} - (a_2^\varepsilon)^2) f^{-N}u, f^{-N}u \rangle + e^{-\lambda t}R_3.$$

We apply once more the sharp Gårding inequality, using (7.2), and we deduce

$$e^{-\lambda t}(\tilde{F}_3 + \tilde{H}_1) \geq -C_3 e^{-\lambda t}t \|f^{-N}u\|_{3/2}^2 - A_3 e^{-\lambda t}t \|f^{-N-3/2}u\|_1^2,$$

where the contribution from  $e^{-\lambda t}R_3$  has been included in the term with the coefficient  $C_3$  in front. On the other hand,

$$t\langle \xi \rangle^3 f^{-2N} \leq t^2 \langle \xi \rangle^3 f^{-2N-1} + t\langle \xi \rangle f^{-2N-3} \leq \delta t^2 \langle \xi \rangle^4 f^{-2N-1} + t\langle \xi \rangle^2 f^{-2N-3} + Q_\delta t^2 f^{-2N-1}. \quad (7.3)$$

Consequently,

$$\begin{aligned} e^{-\lambda t}(\tilde{F}_3 + \tilde{H}_1) &\geq -C_3 \delta t^2 e^{-\lambda t} \|f^{-N-1/2}u\|_2^2 - t(A_3 + C_3) e^{-\lambda t} \|f^{-N-3/2}u\|_1^2 \\ &\quad - C_3 Q_\delta t^2 e^{-\lambda t} \|f^{-N-1/2}u\|_0^2. \end{aligned}$$

The first term on the right hand side can be absorbed by the term  $\frac{t^2}{3} \|\mathbf{a}_2^\varepsilon f^{-N-1/2}u\|_0^2$  in (6.17) choosing  $\delta$  small and exploiting the ellipticity of  $\mathbf{a}_2^\varepsilon$ . The second one can be absorbed by  $\frac{2}{3}N^2 C_1 e^{-\lambda t} \|f^{-N-3/2}u\|_1^2$  taking  $t$  small depending on  $A_3$  and  $C_3$ . Finally, the third term can be absorbed by  $\frac{1}{6}\lambda N e^{-\lambda t} \|f^{-N-2}u\|_0^2$  taking  $\lambda$  large or  $t$  small. We conclude that to absorb the term  $\tilde{F}_3$  we don't need to choose  $N$  in a special way and this is natural since in  $\tilde{F}_3$  we have a coefficient  $t$ .

**7.3. Analysis of  $\tilde{F}_2 + \tilde{F}_{18}$ .** We have

$$\begin{aligned} \tilde{F}_2 + \tilde{F}_{18} &= -\frac{2}{3} \operatorname{Re}\langle f^{-2N} \mathbf{a}_2^\varepsilon u, u'' \rangle + 2 \operatorname{Im}\langle f^{-2N} \mathbf{b}_2^\varepsilon u, u'' \rangle = 2 \operatorname{Im}\langle f^{-2N} (-\frac{1}{3}i\mathbf{a}_2^\varepsilon + \mathbf{b}_2^\varepsilon)u, u'' \rangle \\ &= 2 \operatorname{Im}\langle (-\frac{1}{3}i\mathbf{a}_2^\varepsilon + \mathbf{b}_2^\varepsilon) f^{-N+1/2}u, f^{-N-1/2}u'' \rangle + R_2. \end{aligned}$$

Here

$$R_2 = 2 \operatorname{Im}\langle \left\{ f^{-1/2} \left( [f^{-N+1/2}, (-\frac{1}{3}i\mathbf{a}_2^\varepsilon + \mathbf{b}_2^\varepsilon)] f^{N-1/2} \right) f^{1/2} \right\} f^{-N}u, f^{-N}u'' \rangle$$

and it is easy to absorb this term since the operator  $\left\{ f^{-1/2} \left( \dots \right) f^{1/2} \right\}$  is a first order order pseudodifferential operator. Thus  $R_2$  can be absorbed exactly in the same way as we did above for  $A_2$ .

It is important to note that the subprincipal symbol of the operator  $\mathcal{P}$  for  $t = \tau = 0$  has the form

$$p_2'(0, x, \xi) = -\frac{i}{2}a_2^\varepsilon(0, x, \xi) + b_2^\varepsilon(0, x, \xi).$$

Thus

$$-\frac{1}{3}ia_2^\varepsilon(0, x, \xi) + b_2^\varepsilon(0, x, \xi) = \left[ \frac{1}{6}i + \frac{p_2'(0, x, \xi)}{a_2^\varepsilon(0, x, \xi)} \right] a_2^\varepsilon(0, x, \xi).$$

Let us introduce the number

$$\Pi = \frac{1}{3} + \max_{x \in \bar{U}, |\xi|=1} \left| \frac{p_2'(0, x, \xi)}{a_2^\varepsilon(0, x, \xi)} \right|.$$

Here  $U \subset \mathbb{R}^n$  is the open set in the space variables where we are studying the Cauchy problem. Without loss of generality we may assume that the symbols  $a_2$  and  $b_2$  are homogeneous of order 2 with respect to  $\xi$  (otherwise we must take the max over  $\xi \in \mathbb{R}^n$ ). Thus we have

$$\begin{aligned} |\tilde{F}_2 + \tilde{F}_{18}| &\leq 2 \left| \operatorname{Im} \langle [-\frac{1}{3}ia_2^\varepsilon + b_2^\varepsilon] f^{-N+1/2}u, f^{-N-1/2}u'' \rangle \right| + |R_2| \\ &\leq 2 \left| \operatorname{Im} \langle \left[ \frac{1}{6}i + \frac{p_2'}{a_2^\varepsilon}(0, x, D_x) \right] a_2^\varepsilon(0, x, D_x) f^{-N+1/2}u, f^{-N-1/2}u'' \rangle \right| + |R_2| + |R_3| \\ &\leq \frac{1}{(1+\alpha)\sqrt{3}\Pi} \left\| \left[ \frac{1}{6}i + \frac{p_2'}{a_2^\varepsilon}(0, x, D_x) \right] a_2^\varepsilon(0, x, D_x) f^{-N+1/2}u \right\|_0^2 \\ &\quad + (1+\alpha)\sqrt{3}\Pi \|f^{-N-1/2}u''\|_0^2 + |R_2| + |R_3| \end{aligned}$$

with  $0 < \alpha < 1/2$ . Here  $R_3$  has a similar form as  $R_2$  and it appear since in the product of the operators

$$\left[ \frac{1}{6}i + \frac{p_2'}{a_2^\varepsilon}(0, x, D_x) \right] a_2^\varepsilon(0, x, D_x)$$

the principal term has symbol given by the product of symbols and we must take into account lower term given by a first order operator.

In the analysis below we omit the terms  $R_2$  and  $R_3$  since they can be estimated as specified above. We **choose**  $N = 3\sqrt{3}\Pi + N_0$ , where the integer  $N_0 \geq 1$  is **independent** of  $\Pi$  and will be chosen later. The term above involving  $u''$  can be absorbed by the corresponding term  $e^{-\lambda t} \frac{N}{2} \|f^{-N-1/2}u''\|_0^2$  since  $\frac{N}{2} = \frac{3}{2}\sqrt{3}\Pi + \frac{N_0}{2}$ .

It remains to study the first term on the right hand side.

Let  $\chi \in C^\infty(\mathbb{R}^n)$  be a function such that  $0 \leq \chi(x) \leq 1$ ,  $\chi(x) = 0$  for  $|x| \leq 1$  and  $\chi(x) = 1$  for  $|x| \geq 2$ . We write

$$a_2^\varepsilon(0, x, \xi) = \chi(\xi\delta)a_2^\varepsilon(0, x, \xi) + (1 - \chi(\xi\delta))a_2^\varepsilon(0, x, \xi)$$

with  $\delta > 0$ . The operator with symbol  $(1 - \chi(\xi\delta))a_2^\varepsilon(0, x, \xi)$  is smoothing and the analysis of the corresponding term is covered by using the argument for lower order terms. On the other hand, the norm in  $\mathcal{L}(L^2(U))$  of zero order operator

$$\left[ \frac{1}{6}i + \frac{p_2'}{a_2^\varepsilon}(0, x, D_x) \right] \chi(D_x\delta) \tag{7.4}$$

is not greater than  $\Pi$  if  $\delta$  is chosen small enough depending on the symbols  $a_2$  and  $p_2'$  (see Theorem 18.1.15 in Hörmander, [6].)

We are going to study the term

$$-\frac{\Pi}{(1+\alpha)\sqrt{3}} \|a_2^\varepsilon f^{-N+1/2}u\|_0^2 = -\frac{\Pi}{(1+\alpha)\sqrt{3}} \left[ \operatorname{Re} \langle (a_2^\varepsilon)^2(0, x, D_x) f^{-N+1/2}u, f^{-N+1/2}u \rangle + |B_3| \right], \tag{7.5}$$

where  $|B_3| \leq C_6 \|f^{-N+1/2}u\|_{3/2}^2$  and  $(a_2^\varepsilon)^2(0, x, D_x)$  means  $\text{Op}((a_2^\varepsilon)^2)$ . We have the inequality

$$(a_2^\varepsilon)^2 \leq t(a_2^\varepsilon)^2 f^{-1} + a_2^\varepsilon \alpha_0 f^{-3} \leq t^2 f^{-1} (a_2^\varepsilon)^2 f^{-1} + 2t f^{-2} a_2^\varepsilon \alpha_0 f^{-2} + f^{-3} \alpha_0^2 f^{-3}.$$

We can apply the sharp Gårding equality and we estimate

$$\begin{aligned} -\text{Re} \langle (a_2^\varepsilon)^2 f^{-N+1/2}u, f^{-N+1/2}u \rangle - |B_3| &\geq -t^2 \text{Re} \langle (a_2^\varepsilon)^2 f^{-N-1/2}u, f^{-N-1/2}u \rangle \\ &\quad - 2t \text{Re} \langle \alpha_0 \mathbf{a}_2^\varepsilon f^{-N-3/2}u, f^{-N-3/2}u \rangle - A_3^2 \|f^{-N-5/2}u\|_0^2 - Y_1 \|f^{-N+1/2}u\|_{3/2}^2 \\ &\geq -t^2 \|\mathbf{a}_2^\varepsilon f^{-N-1/2}u\|_0^2 - 2t \text{Re} \langle \alpha_0 \mathbf{a}_2^\varepsilon f^{-N-3/2}u, f^{-N-3/2}u \rangle - A_3^2 \|f^{-N-5/2}u\|_0^2 \\ &\quad - Y_2 \|f^{-N+1/2}u\|_{3/2}^2 = \sum_{j=1}^4 \Gamma_j. \end{aligned}$$

Here we have used the fact that

$$t^2 \text{Re} \langle (a_2^\varepsilon)^2 f^{-N-1/2}u, f^{-N-1/2}u \rangle = t^2 \|\mathbf{a}_2^\varepsilon f^{-N-1/2}u\|_0^2 + B_4,$$

where  $|B_4| \leq A_4 t^2 \|f^{-N-1/2}u\|_{3/2}^2$ . Since  $t^2 f^{-2} \leq 9$ , we included the term  $B_4$  in the above sum taking  $Y_2 \geq Y_1$ . Notice that  $Y_1, Y_2$  depend only on the symbol  $a_2^\varepsilon(0, x, \xi)$ .

It is convenient to transform the term  $\Gamma_4$ . For this purpose we use the inequality

$$\begin{aligned} \langle \xi \rangle^3 f^{-2N+1} &\leq t \langle \xi \rangle^3 f^{-2N} + \langle \xi \rangle f^{-2N-2} \leq t^2 \langle \xi \rangle^3 f^{-2N-1} + t \langle \xi \rangle f^{-2N-3} + \langle \xi \rangle^2 f^{-2N-2} \\ &\leq \delta_1 t^2 \langle \xi \rangle^4 f^{-2N-1} + D_{\delta_1} t^2 f^{-2N-1} + 2t \langle \xi \rangle^2 f^{-2N-3} + f^{-2N-5}. \end{aligned}$$

Thus

$$\begin{aligned} \|f^{-N+1/2}u\|_{3/2}^2 &\leq \delta_1 t^2 \|f^{-N-1/2}u\|_2^2 + 2t \|f^{-N-3/2}u\|_1^2 + \|f^{-N-5/2}u\|_0^2 + D_{\delta_1} t^2 \|f^{-N-1/2}u\|_0^2 \\ &\leq \delta_1 C^2 t^2 \|\mathbf{a}_2^\varepsilon f^{-N-1/2}u\|_0^2 + 2t \|f^{-N-3/2}u\|_1^2 + \|f^{-N-5/2}u\|_0^2 + D'_{\delta_1} t^2 \|f^{-N-1/2}u\|_0^2. \end{aligned}$$

We take  $\delta_1 > 0$  small enough so that  $\delta_1 C^2 Y_2 \leq \delta < \alpha$  and we couple the term with the factor  $t^2$  with that also involving  $t^2$  in  $\Gamma_1$ . Next we fix  $\delta_1$  and for small  $t$  we have  $D'_{\delta_1} Y_2 t^2 \|f^{-N-1/2}u\|_0^2 \leq \|f^{-N-5/2}u\|_0^2$ . We sum this term with  $\Gamma_3$ . Consequently, we get

$$\begin{aligned} \sum_{j=1}^4 \Gamma_j &\geq -(1 + \delta) t^2 \|\mathbf{a}_2^\varepsilon f^{-N-1/2}u\|_0^2 - 2t (C_{a_2} + Y_2) \|f^{-N-3/2}u\|_1^2 \\ &\quad - (A_3^2 + 2Y_2) \|f^{-N-5/2}u\|_0^2 \\ &= \sum_{j=1}^3 Z_j. \end{aligned} \tag{7.6}$$

Here we have used that

$$\text{Re} \langle \mathbf{a}_2^\varepsilon f^{-N-3/2}u, f^{-N-3/2}u \rangle \leq C_{a_2} \|f^{-N-3/2}u\|_1^2$$

and the constant  $C_{a_2}$  depends on  $a_2^\varepsilon(0, x, \xi)$ , while  $A_3 = \|\alpha_0^\varepsilon(0, x, D_x)\|_{L^2(U) \rightarrow L^2(U)}$ .

Now our next task is to **absorb** the sum

$$\frac{\Pi}{(1 + \alpha)\sqrt{3}} \sum_{j=1}^3 Z_j.$$

To treat  $\frac{\Pi}{(1+\alpha)\sqrt{3}}Z_1$ , we exploit the fraction  $\frac{N}{9}t^2\|\mathbf{a}_2^\varepsilon f^{-N-1/2}u\|_0^2$  of the term  $\frac{N}{3}t^2\|\mathbf{a}_2^\varepsilon f^{-N-1/2}u\|_0^2$  in (6.17). Thus we get

$$\left(-\frac{(1+\delta)\Pi}{(1+\alpha)\sqrt{3}} + \frac{N}{9}\right)t^2\|\mathbf{a}_2^\varepsilon f^{-N-1/2}u\|_0^2 > \frac{N_0}{9}t^2\|\mathbf{a}_2^\varepsilon f^{-N-1/2}u\|_0^2$$

since  $0 < \delta < \alpha$ .

To handle  $Z_2$ , we must absorb  $-\frac{\Pi}{(1+\alpha)\sqrt{3}}2t(C_{a_2} + Y_2)e^{-\lambda t}\|f^{-N-3/2}u\|_1^2$ . For this purpose we use the term with  $t\|f^{-N-3/2}u\|_1^2$  in (6.17) and the inequality

$$C_1N^2 > \frac{\sqrt{3}\Pi}{(1+\alpha)}(C_{a_2} + Y_2).$$

Since  $N > 3\sqrt{3}\Pi$ , this is possible choosing  $N$  so that  $3NC_1 \geq \frac{C_{a_2}+Y_2}{(1+\alpha)}$ . Notice that the constants  $C_1, C_{a_2}$  and  $Y_2$  depend only on the symbols of  $a_2(0, x, \xi)$ .

For  $Z_3$  we use the corresponding term in (6.17) and we choose  $N$  large to arrange

$$\frac{1}{2}N(2N + 3 - 2C_1) \geq \frac{\sqrt{3}\Pi}{(1+\alpha)}(A_3 + 2Y_2).$$

Finally we conclude that to deal with the terms  $\tilde{F}_1, \tilde{F}_2, \tilde{F}_3, \tilde{F}_{18}$ , it suffices to take  $N = 3\sqrt{3}\Pi + N_0$  with a constant  $N_0$  depending only on the symbol  $a_2(0, x, \xi)$ . Moreover, in (6.17) we have exploited only one third of the term involving  $t^2$  and we still have

$$\frac{2N}{9}t^2e^{-\lambda t}\|\mathbf{a}_2^\varepsilon f^{-N-1/2}u\|_0^2 + \frac{N_0}{2}e^{-\lambda t}\|f^{-N-1/2}\partial_t^2u\|_0^2$$

available for further use.

**7.4. Analysis of  $\tilde{F}_{19}$ .** We have

$$\tilde{F}_{19} = \frac{2}{3}t \operatorname{Im}\langle f^{-2N}\mathbf{b}_2^\varepsilon u, \mathbf{a}_2^\varepsilon u \rangle$$

and we get

$$\begin{aligned} \tilde{F}_{19} &= \frac{2}{3}t \operatorname{Im}\langle f^{-2N}(\mathbf{b}_2^\varepsilon - \frac{i}{2}\mathbf{a}_2^\varepsilon)u, \mathbf{a}_2^\varepsilon u \rangle + \frac{1}{3}t \operatorname{Re}\langle f^{-2N}\mathbf{a}_2^\varepsilon u, \mathbf{a}_2^\varepsilon u \rangle \\ &= \frac{2}{3}t \operatorname{Im}\langle \frac{p_2'}{a_2^\varepsilon}(0, x, D_x)\mathbf{a}_2^\varepsilon f^{-N+1/2}u, \mathbf{a}_2^\varepsilon f^{-N-1/2}u \rangle \\ &\quad + \frac{1}{3}t\|\mathbf{a}_2^\varepsilon f^{-N}u\|_0^2 + R_4 \\ &= \frac{2}{3}\tilde{H}_{19} + \frac{1}{3}t\|\mathbf{a}_2^\varepsilon f^{-N}u\|_0^2 + R_4, \end{aligned}$$

where  $R_4$  contains lower order terms which can be treated as above. We must deal with the term  $\frac{2}{3}\tilde{H}_{19}$ . Using the notation of the preceding subsection and arguing as we did for the operator on (7.4), modulo easy-to-estimate errors we have

$$\frac{2}{3}\tilde{H}_{19} \geq -\frac{\Pi}{\mu}\|\mathbf{a}_2^\varepsilon f^{-N+1/2}u\|_0^2 - \frac{\mu\Pi}{9}t^2\|\mathbf{a}_2^\varepsilon f^{-N-1/2}u\|_0^2.$$

We choose  $\mu = 3(\sqrt{3} - \sqrt{2-\delta}) > 0$ , where  $0 < \delta < 1/2$  is the constant in the previous subsection we apply to the first term above the same argument we did for (7.5) leading to (7.6).

To absorb the term with a factor  $t^2$ , we take into account the term with the factor  $t^2$  that we have available, as specified at the end of section 7.3, the term with  $-(1+\delta)t^2$  in  $Z_1$  multiplied by  $\frac{\Pi}{\mu}$  and term with coefficient  $\frac{\mu\Pi}{9}t^2$  in the last estimate for  $\tilde{H}_{19}$ . Since  $\frac{2}{9}N = \frac{2\sqrt{3}}{3}\Pi + \frac{2}{9}N_0$ , summing all terms involving  $t^2\Pi\|\mathbf{a}_2^\varepsilon f^{-N-1/2}u\|_0^2$ , we get

$$t^2 \left[ \frac{2}{3}\sqrt{3} - \frac{(1+\delta)}{\mu} - \frac{\mu}{9} \right] \Pi \|\mathbf{a}_2^\varepsilon f^{-N-1/2}u\|_0^2 = 0.$$

The analysis of other terms in (7.6) multiplied by  $\frac{\Pi}{\mu}$  is the same and we must increase only  $N_0$  if it is necessary. Thus the term  $\tilde{F}_{19}$  can be absorbed. We should mention that after the analysis of  $\tilde{F}_2, \tilde{F}_{18}$  and  $\tilde{F}_{19}$  we have in our disposition the positive terms

$$\frac{N-4}{3} \operatorname{Re} \langle \mathbf{a}_2^\varepsilon f^{-N-1/2}u', f^{-N-1/2}u' \rangle + N_0 e^{-\lambda t} \left( \frac{1}{2} \|f^{-N-1/2}u''\|_0^2 + \frac{1}{3} t^2 \|\mathbf{a}_2^\varepsilon f^{-N-1/2}u\|_0^2 \right). \quad (7.7)$$

**7.5. Analysis of  $F_4, F_5, G_1$  and  $G_2$ .** We have

$$F_4 + F_5 = 2 \operatorname{Re} \langle f^{-N}u'', f^{-N}(t\tilde{\alpha}_1^\varepsilon + \alpha_1^\varepsilon)u \rangle = 2 \operatorname{Re} \langle f^{-N}u'', \beta_1 f^{-N}u \rangle$$

with a first order operator  $\beta_1$ . Thus

$$e^{-\lambda t} |F_4 + F_5| \leq \delta^{-1} e^{-\lambda t} \|f^{-N}u''\|_0^2 + C_8 \delta e^{-\lambda t} \|f^{-N}u\|_1^2.$$

The first term on the right can be absorbed by  $\lambda e^{-\lambda t} E_N(u)$  taking  $\lambda$  large, while the second one can be absorbed by  $N \frac{C_1}{3} \lambda e^{-\lambda t} \|f^{-N-1}u\|_1^2$  choosing also  $\lambda$  large. The term  $G_1$  can be absorbed by  $\lambda e^{-\lambda t} E_N(u)$ , while  $G_2$  can be absorbed by the positive term involving  $t^2$  in (7.7) and other positive terms in (6.17).

Now it remains to treat the terms which have coefficient  $\varepsilon^{1/3}$  or  $\varepsilon^{2/3}$ .

**7.6. Analysis of  $G_j, j = 3, \dots, 6$ .** The term  $G_3 = 2Nt\varepsilon^{2/3}A_{N+1/2}^{(2)}(u)$  is a sum of terms. They can be estimated following our previous arguments but we have an advantage since we have a factor  $t\varepsilon^{2/3}$ . Consider a typical term

$$L_4 = 2Nt\varepsilon^{2/3} \operatorname{Re} \langle f^{-2N-1}\tilde{a}_2^\varepsilon u, u' \rangle.$$

We have modulo lower order terms

$$|L_4| \leq N\varepsilon^{2/3} \left( \|f^{-N-1/2}u''\|_0^2 + t^2 C_{a_2} \|f^{-N-1/2}u\|_2^2 \right)$$

and we may absorb this term by (7.7) taking  $\varepsilon$  small to arrange  $N\varepsilon^{2/3} \leq 1$ ,  $N\varepsilon^{2/3}C_{a_2} \leq 1$  and  $N_0$  large if it necessary. Let us consider the term

$$L_3 = 2Nt^2\varepsilon^{2/3} \operatorname{Re} \langle f^{-2N-1}\tilde{a}_2^\varepsilon u', u' \rangle.$$

Modulo lower order terms we have

$$|L_3| \leq Nt^2\varepsilon^{2/3}C_{a_2} \|f^{-N-1/2}u'\|_1^2$$

and this can be absorbed taking  $N\varepsilon^{2/3}C_{a_2} \leq 1$  and  $t$  small.

Now we pass to  $G_4$  and we consider a typical term

$$L_5 = 2Nt^2\varepsilon^{1/3}2 \operatorname{Im} \langle f^{-2N-1}a_3^\varepsilon u, u' \rangle.$$



Modulo lower order term appearing when we commute  $a_3^\varepsilon$  and  $f^{-N-1/2}$ , we must absorb

$$-2N\varepsilon^{1/3} \left[ C_{a_3} t^3 \|f^{-N-1/2} u\|_2^2 + t \|f^{-N-1/2} u\|_1^2 \right].$$

The second term can be absorbed by  $\frac{1}{6}\lambda N C_1 t \|f^{-N-1} u\|_1^2$  taking  $\varepsilon$  small or  $\lambda$  large. The first one can be absorbed by (7.7) taking  $\varepsilon$  small or  $t$  small. The analysis of

$$L_6 = -\frac{2}{3} N t^2 \varepsilon^{1/3} 2 \operatorname{Im} \langle f^{-2N-1} a_1^\varepsilon a_2^\varepsilon u, u' \rangle$$

is completely similar since we have a third order operator  $a_1^\varepsilon a_2^\varepsilon$ . In this way we may treat all other terms in  $G_j$ .

**7.7. Analysis of  $F_6$  and  $F_7$ .** We have

$$F_6 = -\frac{2}{3} \varepsilon^{2/3} t^2 \operatorname{Re} \langle f^{-2N} (\partial_t a_2^\varepsilon) u, a_2^\varepsilon u \rangle.$$

We have to estimate

$$-e^{2/3} \left[ \delta C_8 t^2 \|f^{-N} u\|_2^2 + \delta^{-1} t^2 \|a_2^\varepsilon f^{-N} u\|_0^2 \right]$$

To do this we take  $\delta$  small to arrange  $\delta C_8 \leq 1$  and next we exploit the terms in  $\lambda E_N(u)$  with  $t^2$  choosing  $\lambda$  large et  $\varepsilon$  petit.

On the other hand,

$$\begin{aligned} F_7 &= \varepsilon^{1/3} t \, 2 \operatorname{Re} \langle f^{-2N} \alpha_0^\varepsilon u'', u'' \rangle \\ &= 2\varepsilon^{1/3} t \operatorname{Re} \langle (\alpha_0^\varepsilon + [f^{-N}, \alpha_0^\varepsilon] f^N) f^{-N} u'', f^{-N} u'' \rangle \\ &= 2\varepsilon^{1/3} t \operatorname{Re} \langle \beta_0^\varepsilon f^{-N} u'', f^{-N} u'' \rangle, \end{aligned}$$

with a suitable symbol  $\beta_0^\varepsilon$  of order zero. We then immediately get

$$e^{-\lambda t} |F_6| \leq C_{F_6} \varepsilon^{2/3} t e^{-\lambda t} \|f^{-N} u''\|_0^2,$$

which is absorbed directly by the second time derivative in  $e^{-\lambda t} \lambda E_N(u)$ .

The analysis of other  $F_j$  terms is simpler since we have  $t$  or  $t^2$ , powers of  $\varepsilon$  and no factor  $N$ . Taking  $\varepsilon$  small we may absorb all these terms. Since the analysis is quite similar to that exposed above, we leave the details to the reader. This completes the estimate of the errors.

As a consequence (6.17) can be rewritten as a true energy estimate as

$$\begin{aligned}
e^{-\lambda t} \|f^{-N+\frac{1}{2}} \mathcal{P}u\|^2 &\geq \partial_t \left( e^{-\lambda t} \mathcal{E}_N(u) + \frac{N}{6} e^{-\lambda t} \|f^{-N-1} u'\|_0^2 \right. \\
&+ \frac{N}{6} e^{-\lambda t} \|f^{-N-2} u\|_0^2 + \frac{N}{3} C_1 t e^{-\lambda t} \|f^{-N-1} u\|_1^2 \left. \right) + \lambda K_1 e^{-\lambda t} E_N(u) \\
&+ K_2 e^{-\lambda t} \left[ \|f^{-N-1/2} u''\|_0^2 + t \|f^{-N-1/2} u'\|_1^2 + t^2 \|f^{-N-1/2} u\|_2^2 \right] \\
&+ N^2 K_3 e^{-\lambda t} \left[ \|f^{-N-3/2} u'\|_0^2 + \|f^{-N-5/2} u\|_0^2 + t e^{-\lambda t} \|f^{-N-3/2} u\|_1^2 \right] \\
&+ \lambda N K_4 e^{-\lambda t} \left\{ \|f^{-N-1} u'\|_0^2 + \|f^{-N-2} u\|_0^2 + t \|f^{-N-1} u\|_1^2 \right\}, \tag{7.8}
\end{aligned}$$

where  $K_j$ ,  $j = 1, \dots, 4$ , are suitable positive constants independent of  $\lambda$  and  $N$ .

**7.8. Estimates of the terms on the boundary  $s = T$ .** Assume that  $0 \leq s < T \leq 1$  and  $u = D_t u = D_t^2 u = 0$  when  $t = s$ . We integrate in (7.8) from  $t$  to  $T$  with respect to  $s$ . As a result we obtain integrals  $\int_t^T (\dots) ds$  and for  $s = T$  the following boundary terms

$$\begin{aligned}
e^{-\lambda T} \mathcal{E}_N((u(T, \cdot))) &+ \frac{1}{6} N e^{-\lambda T} \|f^{-N-1} \partial_t u(T, \cdot)\|_0^2 + \frac{1}{6} N e^{-\lambda T} \|f^{-N-2} u(T, \cdot)\|_0^2 \\
&+ \frac{1}{3} N C_1 e^{-\lambda T} \|f^{-N-1} u(T, \cdot)\|_1^2. \tag{7.9}
\end{aligned}$$

Recall that the argument of the previous section yields

$$\mathcal{E}_N(u(T, \cdot)) \geq E_N(u) + T \varepsilon^{2/3} A_N^{(2)}(u(T, \cdot)) + T^2 \varepsilon^{2/3} A_N^{(3)}(u(T, \cdot)).$$

It is clear that  $(T/3 + \langle \xi \rangle^{-2/3})^{-N} \leq c_2 (T/3 + \langle \xi \rangle^{-2/3})^{-N-1}$  with  $c_2 > 0$  independent on  $\xi$  and  $T$ . Thus taking  $\varepsilon$  small we may absorb the negative terms in (7.9) by  $E_N(u)$  and the terms in (7.9) having in their coefficients  $N$  and  $C_1 N$ .

Finally, the contribution of the boundary terms is bounded from below by a positive constant and we may neglect them.

## 8. A PRIORI ESTIMATE

We use the inequalities

$$\begin{aligned}
f^{-1} &= \frac{(1 + |\xi|^2)^{1/3}}{t(1 + |\xi|^2)^{1/3} + 1} \leq \frac{(1 + |\xi|^2)^{1/3}}{1 + t}, \quad t \geq 0, \\
f^{-1} &\geq \frac{1}{1 + t}, \quad 0 \leq t \leq T \leq 1.
\end{aligned}$$

Therefore from (7.8) and the analysis in the Section 7 we deduce for  $\lambda \geq \lambda_0$  the estimate

$$\begin{aligned}
\lambda \int_t^T e^{-\lambda s - 2N \log(1+s)} &\left( \sum_{k=0}^2 s^{2-k} \|\partial_t^k u(s, \cdot)\|_{(2-k)}^2 + \sum_{k=0}^1 \|\partial_t^k u(s, \cdot)\|_{(1-k)}^2 \right) ds \\
&\leq C_0 \int_t^T e^{-\lambda s - 2N \log(1+s)} (1 + s) \|\mathcal{P}u(s, \cdot)\|_{(2N/3-1/3)}^2 ds, \tag{8.1}
\end{aligned}$$

where  $\|\cdot\|_{(m)}$  is the  $H_{(m)}$  norm in  $\mathbb{R}^n$  for fixed  $m$ . Thus we get

$$\begin{aligned} \lambda \int_t^T e^{-\lambda s - 2N(1+s)} \left( \sum_{k=0}^2 s^{2-k} \|\partial_t^k u(s, \cdot)\|_{(2-k)}^2 + \sum_{k=0}^1 \|\partial_t^k u(s, \cdot)\|_{(1-k)}^2 \right) ds \\ \leq C_0 \int_t^T e^{-\lambda s - 2N \log(1+s)} (1+s) \|\mathcal{P}u(s, \cdot)\|_{(2N/3-1/3)}^2 ds. \end{aligned} \quad (8.2)$$

**Remark 8.1.** It is not useful to use the estimate  $f^{-1} \leq \frac{1}{t}$  to bound the term  $\|f^{-N+1/2} \mathcal{P}u\|^2$  in (7.8). If we did then, in (8.1), we would have the integral

$$\int_t^T e^{-\lambda s} \frac{1}{s^{2N-1}} \|\mathcal{P}u(s, \cdot)\|_{(0)}^2 ds$$

and as  $t \rightarrow 0$  this would produce no uniform estimates with respect to  $t \geq 0$ .

**Remark 8.2.** It is possible to apply, on the left hand side of (8.2), the estimate

$$e^{-\lambda s - N \log(1+s)} \geq e^{-2\lambda s}, \quad 0 \leq s \leq T$$

by choosing  $\lambda \geq N \geq \frac{N}{1+s}$  since for such  $\lambda$  we have  $\lambda s - N \log(1+s) \geq 0$ .

To estimate the high order derivatives with respect to  $x$ , consider the operator  $(1 + |D_x|^2)^{\frac{1}{2}p} = \Lambda_p$ ,  $p > 0$  and write

$$\begin{aligned} \Lambda_p \mathcal{P}u = D_t^3(\Lambda_p u) - t \left( a_2 + [\Lambda_p, a_2] \Lambda_p^{-1} \right) D_t(\Lambda_p u) + \left( b_2 + [\Lambda_p, b_2] \Lambda_p^{-1} \right) (\Lambda_p u) \\ + t \left( a_1 + [\Lambda_p, a_1] \Lambda_p^{-1} \right) D_t^2(\Lambda_p u) + \dots \end{aligned}$$

Moreover, we observe that the “perturbations”  $[\Lambda_p, a_2] \Lambda_p^{-1}$ ,  $[\Lambda_p, b_2] \Lambda_p^{-1}$ ,  $[\Lambda_p, a_1] \Lambda_p^{-1}$  have order lower than the terms  $a_2, b_2$  and  $a_1$ , respectively. Then  $v = \Lambda_p u$  satisfies an equation of the type studied above and, moreover,  $(D_t^2 v)(t, x) = (D_t v)(t, x) = v(t, x) = 0$ . Going back to the differential operator  $P$ , we get the following

**Theorem 8.1.** Assume that  $(D_t^2 u)(t, x) = (D_t u)(t, x) = u(t, x) = 0$  and let  $0 \leq t \leq s \leq T$  with a small  $T > 0$ . Then for every  $p \in \mathbb{R}$  there exist  $\Lambda_p$  and a constant  $C_p$  so that for  $\lambda \geq \Lambda_p$ ,  $N = 3\sqrt{3}\Pi + N_0$  and  $u \in C_0^\infty(\mathbb{R}^{n+1})$  we have the estimate

$$\begin{aligned} \lambda \int_t^T e^{-\lambda s - 2N \log(1+s)} \left( \sum_{k=0}^2 s^{2-k} \|\partial_t^k u(s, \cdot)\|_{(2-k+p)}^2 + \sum_{k=0}^1 \|\partial_t^k u(s, \cdot)\|_{(1-k+p)}^2 \right) ds \\ \leq C_p \int_t^T e^{-\lambda s - 2N \log(1+s)} (1+s) \|\mathcal{P}u(s, \cdot)\|_{(2N/3-1/3+p)}^2 ds. \end{aligned} \quad (8.3)$$

Next we discuss the estimates for functions  $u(t, x)$  satisfying the boundary conditions  $D_t^2 u(T, x) = D_t u(T, x) = u(T, x) = 0$  for  $T > 0$ . To do this, we proceed along the same lines as above. We apply to  $\mathcal{P}u$  the operator  $e^{\lambda t} f^{2N}(t, D_x)$  and consider

$$2 \operatorname{Im} \langle e^{\lambda t} f^{2N}(t, D_x) \mathcal{P}u, Mu \rangle.$$

Notice that we have changed  $-\lambda$  to  $\lambda$ ,  $f^{-2N}$  to  $f^{2N}$  and we have a  $+$  sign in front of the scalar product. We then handle the terms in the same way as we did in Sections 5-8. For example,

$$2 \operatorname{Im} \frac{1}{i} \langle e^{\lambda t} f^{2N} \partial_t^3 u, \partial_t^2 u \rangle = -2 \operatorname{Re} \langle e^{\lambda t} f^{2N} \partial_t^3 u, \partial_t^2 u \rangle$$

$$= -\partial_t \left( e^{\lambda t} \|f^{2N} \partial_t^2 u\|^2 \right) + 2Ne^{\lambda t} \|f^N \partial_t^2 u\|^2 + \lambda e^{\lambda t} \|f^{N-\frac{1}{2}} \partial_t^2 u\|^2.$$

Thus we obtain an analog of (6.17) in the form

$$\begin{aligned} e^{\lambda t} \|f^{N+\frac{1}{2}} \mathcal{P}u\|^2 &\geq \partial_t \left( e^{-\lambda t} \mathcal{E}_{-N}(u) + \frac{N}{6} e^{\lambda t} \|f^{-N-1} u'\|_0^2 \right. \\ &\quad \left. + \frac{N}{6} e^{\lambda t} \|f^{N-2} u\|_0^2 + \frac{N}{3} C_1 t e^{\lambda t} \|f^{N-1} u\|_1^2 \right) + \lambda e^{-\lambda t} E_{-N}(u) \\ &\quad + Ne^{\lambda t} \left[ \frac{1}{2} \|f^{N-1/2} u''\|_0^2 + \frac{1}{3} t \operatorname{Re} \langle \mathbf{a}_2^\varepsilon f^{N-1/2} u', f^{N-1/2} u' \rangle + \frac{1}{3} t^2 \|\mathbf{a}_2^\varepsilon f^{N-1/2} u\|_0^2 \right] \\ &\quad + \frac{N^2}{3} e^{\lambda t} \left[ \|f^{N-3/2} u'\|_0^2 + \frac{2N+3-2C_1}{2N} \|f^{N-5/2} u\|_0^2 + 2C_1 t e^{-\lambda t} \|f^{N-3/2} u\|_1^2 \right] \\ &\quad + \frac{1}{3} N C_1 t e^{\lambda t} \|f^{-N-1/2} u'\|_1^2 \\ &\quad + \lambda N e^{\lambda t} \left\{ \frac{1}{12} \|f^{N-1} u'\|_0^2 + \frac{1}{6} \|f^{N-2} u\|_0^2 + \frac{1}{6} C_1 t \|f^{N-1} u\|_1^2 \right\} \\ &\quad + e^{\lambda t} \sum_{j=1}^{19} F'_j + e^{\lambda t} \sum_{j=1}^6 G'_j + \text{lower order terms}, \end{aligned} \quad (8.4)$$

where  $G'_j, F'_j$  are obtained from the corresponding terms  $G_j, F_j$  changing  $N$  by  $-N$ . We repeat the argument of the previous sections and we integrate with respect to  $s$  from  $t$  to  $T$ , assuming  $0 \leq t < T \leq 1$ . Thus we obtain an a priori estimate involving the “weights”  $f^{2N-k}$ ,  $-1 \leq k \leq 5/2$ .

On the other hand,

$$\begin{aligned} f^{2N+1} &\leq (t+1)^{2N+1}, \quad 0 \leq t < T \leq 1, \\ f^{2N} &\geq (1+t)^{2N} (1+|\xi|^2)^{-2N/3}. \end{aligned}$$

Consequently, for  $\lambda \geq \lambda_0 > 0$ , we deduce

$$\begin{aligned} \lambda \int_t^T e^{\lambda s + 2N \log(1+s)} \left( \sum_{k=0}^2 s^{2-k} \|\partial_t^k u(s, \cdot)\|_{(2-k-2N/3)}^2 + \sum_{k=0}^1 \|\partial_t^k u(s, \cdot)\|_{(1-k-2N/3)}^2 \right) ds \\ \leq C_0 \int_t^T e^{\lambda s + 2N \log(1+s)} (1+s) \|\mathcal{P}u(s, \cdot)\|_{(0)}^2 ds. \end{aligned} \quad (8.5)$$

Finally, we may make a shift in the Sobolev indices for this estimate and consider  $\|\cdot\|_{(p)}$  norms. Thus we eventually obtain the following

**Theorem 8.2.** *Assume that  $(D_t^2 u)(T, x) = (D_t u)(T, x) = u(T, x) = 0$  and let  $0 \leq t \leq s \leq T$  with a small  $T > 0$ . Then for every  $p \in \mathbb{R}$  there exist  $\Lambda_p$  and a constant  $C_p$  so that for  $\lambda \geq \Lambda_p$ ,  $N = 3\sqrt{3}\Pi + N_0$  and  $u \in C_0^\infty(\mathbb{R}^{n+1})$  we have the estimate*

$$\begin{aligned} \lambda \int_t^T e^{\lambda s + 2N \log(1+s)} \left( \sum_{k=0}^2 s^{2-k} \|\partial_t^k u(s, \cdot)\|_{(2-k+p)}^2 + \sum_{k=0}^1 \|\partial_t^k u(s, \cdot)\|_{(1-k+p)}^2 \right) ds \\ \leq C_p \int_t^T e^{\lambda s + 2N \log(1+s)} (1+s) \|Pu(s, \cdot)\|_{(2N/3+p)}^2 ds. \end{aligned} \quad (8.6)$$

For the local uniqueness result it is more convenient to have estimates for the operator  $\mathcal{P}^*$ , the adjoint to  $\mathcal{P}$ . We have

$$\mathcal{P}^* = D_t^3 u + ta_1 D_t^2 - ta_2 D_t + t^2 a_3 + \bar{b}_2 + ia_2 + t\alpha_2 + \text{lower order terms},$$

where  $\alpha_2$  is a second order operator with respect to  $x$ . The subprincipal symbol of  $\mathcal{P}^*$  for  $t = \tau = 0$  has the form

$$\frac{i}{2}a_2(0, x, \xi) + \bar{b}_2(0, x, \xi) = \overline{p'_2}(0, x, \xi).$$

Thus the number  $\Pi^*$  corresponding to  $\mathcal{P}^*$  coincides with  $\Pi$  and Theorems 8.1 and 8.2 hold for the operator  $\mathcal{P}^*$  changing, if it is necessary,  $\Lambda_p$  and  $C_p$ .

Applying Theorems 8.1 for  $P$  and Theorem 8.2 for  $P^*$ , we can establish an existence and uniqueness results for the Cauchy problem in  $\{(t, x) : 0 \leq t \leq T, x \in U \text{ with sufficiently small } T\}$ . To fix the notations, we say that  $f \in H_{(q,s)}^{loc}(G)$  if  $\varphi f \in H_{(q,s)}(G)$  for all  $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$  and  $g \in H_{(q,s)}(\mathbb{R}^{n+1})$  if

$$\|g\|_{(q,s)}^2 = (2\pi)^{-(n+1)} \int (1 + |\tau|^2)^q (1 + |\xi|^2)^s |\hat{g}(\tau, \xi)|^2 d\tau d\xi < \infty.$$

Since  $P$  is strictly hyperbolic for  $0 < t \leq T$ , we may repeat with minor modifications the proof of Theorem 23.4.5 in [6] to obtain the following

**Theorem 8.3.** *Let  $P$  be a differential operators with  $C^\infty$  coefficients in  $G = [0, T] \times U$  satisfying the hypothesis  $(H_0) - (H_2)$  and let  $Y \subset\subset U$ . For  $T$  sufficiently small and for  $f \in H_{(0,s)}^{loc}(G)$  having support in  $\bar{G}$  one can find an unique  $u \in H_{(2,s+2-2N/3)}^{loc}(G)$  with support in  $\bar{G}$  so that  $Pu = f$  in  $(0, T) \times Y$ .*

We leave the details to the reader.

In conclusion the conjecture for strongly hyperbolic operators with triple characteristics is true for operators satisfying  $(H_0) - (H_2)$ .

It is an open problem to study effectively hyperbolic operators having triple characteristics only at some points on the hyperplane  $t = 0$ . For this purpose probably some microlocal models should be examined and furthermore one should construct a more sophisticated “time function” possibly related to the microlocal form of the operator, as in [13]. On the other hand, finding a microlocal form for the discriminant  $\Delta$  of the cubic equation  $p_3(t, x, \tau, \xi) = 0$  in a neighborhood of a triple characteristic point is without any doubt a major difficulty.

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